PERIODICITIES OF T AND Y-SYSTEMS, DILOGARITHM IDENTITIES, AND CLUSTER ALGEBRAS I: TYPE B_r

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ABSTRACT. We prove the periodicities of the restricted T and Y-systems associated with the quantum affine algebra of type B_r at any level. We also prove the dilogarithm identities for the Y-systems of type B_r at any level. Our proof is based on the tropical Y-systems and the categorification of the cluster algebra associated with any skew-symmetric matrix by Plamondon. Using this new method, we also give an alternative and simplified proof of the periodicities of the T and Y-systems associated with pairs of simply laced Dynkin diagrams.

CONTENTS

1. MAIN RESULTS

1.1. **Background.** The T and Y-systems are systems of algebraic relations originally associated with *quantum affine algebras* [KNS2, Nkj, Her1], or more generally, with the quantum affinizations of a wide class of quantum Kac-Moody algebras [Her2, KNS3].

On the other hand, these T and Y-systems also appear naturally in *cluster algebras* [FZ2, FZ3]. This identification has provided several fruitful results. The periodicities of Y-systems were proved by [FZ2] for any simply laced type at level 2 (in our terminology). Here we mean by 'simply laced' the Y-systems associated with the *quantum affine algebras of simply laced type*. The periodicities of Y-systems were further proved by [Ke1, Ke2] for any simply laced type at any level, by the combination with the *cluster category* method. Using the method of [Ke1, Ke2], the periodicities of T-systems were also proved by [IIKNS] for any simply laced type at any level. Closely related to the Y-systems, the dilogarithm identities were proved

by [C] for any simply laced type at level 2 based on the result of [FZ2], and further proved by [Nkn] for any simply laced type and any level. So far, however, all these systematic treatments were limited to the simply laced case only, since the above methods are not straightforwardly applicable to the nonsimply laced case.

In this paper and the subsequent one [IIKKN], we prove the periodicities of T and Y-systems, and also the dilogarithm identities, in the *nonsimply laced* case using the cluster algebra/cluster category method with suitable modifications to the method used in the simply laced case. We remark that the nonsimply laced systems here are different from *another* class of nonsimply laced systems arising from cluster algebras and studied in [FZ2, FZ3].

As is often the case in the nonsimply laced setting, each type requires some nonuniform, 'customized' treatment. So, in this paper, we concentrate on type B_r and highlight the underlying common method. Then, separately in [IIKKN], types C_r, F_4, and G_2 will be treated with emphasis on the special features of each case.

1.2. **Restricted T and Y-systems of type B_r.** Let B_r be the Dynkin diagram of type B with rank r, and $I = \{1, \ldots, r\}$ be the enumeration of the vertices of B_r as below.

$$\underset{1}{\circ}\!-\!\!-\!\underset{2}{\circ}\!-\!\!-\quad\cdots\quad-\!\!-\underset{r-1}{\circ}\!\!\Rrightarrow\!\!\underset{r}{\circ}$$
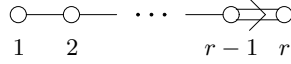
Let $h = 2r$ and $h^\vee = 2r - 1$ be the Coxeter number and the dual Coxeter number of B_r, respectively. We set numbers t_a $(a \in I)$ by

$$(1.1) \qquad t_a = \begin{cases} 1 & a = 1, \ldots, r-1, \\ 2 & a = r. \end{cases}$$

For a given integer $\ell \geq 2$, we introduce a set of triplets (a, m, u),

$$(1.2) \qquad \mathcal{I}_\ell = \mathcal{I}_\ell(B_r) := \{(a, m, u) \mid a \in I; m = 1, \ldots, t_a\ell - 1; u \in \frac{1}{2}\mathbb{Z}\}.$$

Definition 1.1 ([KNS2])**.** Fix an integer $\ell \geq 2$. The *level ℓ restricted T-system* $\mathbb{T}_\ell(B_r)$ *of type B_r (with the unit boundary condition)* is the following system of relations (1.3) for a family of variables $T_\ell = \{T_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell\}$, where $T_m^{(0)}(u) = T_0^{(a)}(u) = 1$, and furthermore, $T_{t_a\ell}^{(a)}(u) = 1$ (the *unit boundary condition*) if they occur in the right hand sides in the relations:

(Here and throughout the paper, $2m$ (resp. $2m + 1$) in the left hand sides, for example, represents elements $2, 4, \ldots$ (resp. $1, 3, \ldots$).)

$$
\begin{aligned}
&T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \\
&\hspace{8cm} (1 \leq a \leq r-2), \\
(1.3) \quad &T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + T_m^{(r-2)}(u)T_{2m}^{(r)}(u), \\
&T_{2m}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(r)}\left(u+\tfrac{1}{2}\right) = T_{2m-1}^{(r)}(u)T_{2m+1}^{(r)}(u) \\
&\hspace{5cm} + T_m^{(r-1)}\left(u-\tfrac{1}{2}\right)T_m^{(r-1)}\left(u+\tfrac{1}{2}\right), \\
&T_{2m+1}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m+1}^{(r)}\left(u+\tfrac{1}{2}\right) = T_{2m}^{(r)}(u)T_{2m+2}^{(r)}(u) + T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u).
\end{aligned}
$$

Definition 1.2 ([KN])**.** Fix an integer $\ell \geq 2$. The *level ℓ restricted Y-system* $\mathbb{Y}_\ell(B_r)$ *of type B_r* is the following system of relations (1.4) for a family of variables $Y_\ell = \{Y_m^{(a)}(u) \mid (a,m,u) \in \mathcal{I}_\ell\}$, where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = Y_{t_a\ell}^{(a)}(u)^{-1} = 0$ if they occur in the right hand sides in the relations:

(1.4)

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}$$
$$(1 \leq a \leq r-2),$$

$$Y_m^{(r-1)}(u-1)Y_m^{(r-1)}(u+1) = \frac{\begin{array}{c}(1+Y_m^{(r-2)}(u))(1+Y_{2m-1}^{(r)}(u))(1+Y_{2m+1}^{(r)}(u))\\ \times(1+Y_{2m}^{(r)}\left(u-\frac{1}{2}\right))(1+Y_{2m}^{(r)}\left(u+\frac{1}{2}\right))\end{array}}{(1+Y_{m-1}^{(r-1)}(u)^{-1})(1+Y_{m+1}^{(r-1)}(u)^{-1})},$$

$$Y_{2m}^{(r)}\left(u-\tfrac{1}{2}\right)Y_{2m}^{(r)}\left(u+\tfrac{1}{2}\right) = \frac{1+Y_m^{(r-1)}(u)}{(1+Y_{2m-1}^{(r)}(u)^{-1})(1+Y_{2m+1}^{(r)}(u)^{-1})},$$

$$Y_{2m+1}^{(r)}\left(u-\tfrac{1}{2}\right)Y_{2m+1}^{(r)}\left(u+\tfrac{1}{2}\right) = \frac{1}{(1+Y_{2m}^{(r)}(u)^{-1})(1+Y_{2m+2}^{(r)}(u)^{-1})}.$$

Let us write (1.3) in a unified manner

(1.5)
$$T_m^{(a)}\left(u-\tfrac{1}{t_a}\right)T_m^{(a)}\left(u+\tfrac{1}{t_a}\right) = T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u) + \prod_{(b,k,v)\in\mathcal{I}_\ell} T_k^{(b)}(v)^{G(b,k,v;a,m,u)}.$$

Define the transposition ${}^tG(b,k,v;a,m,u) = G(a,m,u;b,k,v)$. Then, we have

(1.6) $$Y_m^{(a)}\left(u-\tfrac{1}{t_a}\right)Y_m^{(a)}\left(u+\tfrac{1}{t_a}\right) = \frac{\prod_{(b,k,v)\in\mathcal{I}_\ell}(1+Y_k^{(b)}(v))^{{}^tG(b,k,v;a,m,u)}}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})}.$$

See [IIKNS, KNS3] and references therein on the background of these systems.

1.3. Periodicities.

Definition 1.3. Let $\mathcal{T}_\ell(B_r)$ be the commutative ring over $\mathbb{Z}$ with identity element, with generators $T_m^{(a)}(u)^{\pm 1}$ $((a,m,u) \in \mathcal{I}_\ell)$ and relations $\mathbb{T}_\ell(B_r)$ together with $T_m^{(a)}(u)T_m^{(a)}(u)^{-1} = 1$. Let $\mathcal{T}_\ell^\circ(B_r)$ be the subring of $\mathcal{T}_\ell(B_r)$ generated by $T_m^{(a)}(u)$ $((a,m,u) \in \mathcal{I}_\ell)$.

A *semifield* $(\mathbb{P}, \oplus)$ is an abelian multiplicative group $\mathbb{P}$ endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$ [FZ3, HW].

Definition 1.4. Let $\mathcal{Y}_\ell(B_r)$ be the semifield with generators $Y_m^{(a)}(u)$ $((a,m,u) \in \mathcal{I}_\ell)$ and relations $\mathbb{Y}_\ell(B_r)$. Let $\mathcal{Y}_\ell^\circ(B_r)$ be the multiplicative subgroup of $\mathcal{Y}_\ell(B_r)$ generated by $Y_m^{(a)}(u)$, $1 + Y_m^{(a)}(u)$ $((a,m,u) \in \mathcal{I}_\ell)$. (Here we use the symbol $+$ instead of $\oplus$ for simplicity.)

The first main result of the paper is the periodicities of the T and Y-systems.

Theorem 1.5 (Conjectured by [IIKNS]). *The following relations hold in* $\mathcal{T}_\ell^\circ(B_r)$.
(i) *Half periodicity:* $T_m^{(a)}(u + h^\vee + \ell) = T_{t_a\ell-m}^{(a)}(u)$.
(ii) *Full periodicity:* $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

Theorem 1.6 (Conjectured by [KNS2]). *The following relations hold in* $\mathcal{Y}_\ell^\circ(B_r)$.
(i) *Half periodicity:* $Y_m^{(a)}(u + h^\vee + \ell) = Y_{t_a\ell-m}^{(a)}(u)$.
(ii) *Full periodicity:* $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$.

1.4. **Dilogarithm identities.** Let $L(x)$ be the *Rogers dilogarithm function* [L, Ki2, Zag]

$$(1.7) \qquad L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy \quad (0 \le x \le 1).$$

It is well known that

$$(1.8) \qquad L(0) = 0, \quad L(1) = \frac{\pi^2}{6},$$

$$(1.9) \qquad L(x) + L(1-x) = \frac{\pi^2}{6} \quad (0 \le x \le 1).$$

We introduce the *constant version* of the Y-system.

Definition 1.7. Fix an integer $\ell \ge 2$. The *level ℓ restricted constant Y-system* $\mathbb{Y}_\ell^c(B_r)$ *of type* B_r is the following system of relations (1.10) for a family of variables $Y_\ell^c = \{Y_m^{(a)} \mid a \in I; m = 1, \ldots, t_a\ell - 1\}$, where $Y_m^{(0)} = Y_0^{(a)-1} = Y_{t_a\ell}^{(a)-1} = 0$ if they occur in the right hand sides in the relations:

$$(Y_m^{(a)})^2 = \frac{(1 + Y_m^{(a-1)})(1 + Y_m^{(a+1)})}{(1 + Y_{m-1}^{(a)}{}^{-1})(1 + Y_{m+1}^{(a)}{}^{-1})} \quad (1 \le a \le r-2),$$

$$(Y_m^{(r-1)})^2 = \frac{(1 + Y_m^{(r-2)})(1 + Y_{2m-1}^{(r)})(1 + Y_{2m}^{(r)})^2(1 + Y_{2m+1}^{(r)})}{(1 + Y_{m-1}^{(r-1)}{}^{-1})(1 + Y_{m+1}^{(r-1)}{}^{-1})},$$

$$(1.10) \qquad (Y_{2m}^{(r)})^2 = \frac{1 + Y_m^{(r-1)}}{(1 + Y_{2m-1}^{(r)}{}^{-1})(1 + Y_{2m+1}^{(r)}{}^{-1})},$$

$$(Y_{2m+1}^{(r)})^2 = \frac{1}{(1 + Y_{2m}^{(r)}{}^{-1})(1 + Y_{2m+2}^{(r)}{}^{-1})}.$$

Proposition 1.8. *There exists a unique positive real solution of* $\mathbb{Y}_\ell^c(B_r)$.

Proof. Set $f_m^{(a)} = Y_m^{(a)}/(1 + Y_m^{(a)})$. Then, (1.10) is equivalent to the system of equations [KNS2, Eq. (B.28)]

$$(1.11) \qquad f_m^{(a)} = \prod_{(b,k)} (1 - f_k^{(b)})^{K_{ab}^{mk}}, \quad K_{ab}^{mk} = (\alpha_a|\alpha_b)(\min(t_b m, t_a k) - \frac{mk}{\ell}),$$

where $(\alpha_a|\alpha_b)$ is the invariant bilinear form for the simple Lie algebra of type B_r with normalization $(\alpha_a|\alpha_a) = 2$ for a long root α_a. By elementary transformations, one can show that every principal minor of the matrix K is positive. Therefore, K is positive definite. Also, it is clear that K is symmetric. Then, the theorem by [NK, Section 1] is applicable. $\qquad \square$

The second main result of the paper is the dilogarithm identities conjectured by Kirillov [Ki1, Eq. (7)], properly corrected by Kuniba [Ku, Eqs. (A.1a), (A.1c)],

Theorem 1.9 (Dilogarithm identities). *Suppose that a family of positive real numbers $\{Y_m^{(a)} \mid a \in I; m = 1, \ldots, t_a\ell - 1\}$ satisfies (1.10). Then, we have the identity*

$$(1.12) \qquad \frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a\ell-1} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h^\vee + \ell} - r,$$

where $\mathfrak{g}$ is the simple Lie algebra of type B_r.

The rational number of the first term in the right hand side of (1.12) is the central charge of the *Wess-Zumino-Witten conformal field theory* of type B_r with level ℓ. The rational number in the right hand side of (1.12) itself is also the central charge of the *parafermion conformal field theory* of type B_r with level ℓ. See [KNS1, Ki2, Nah, Zag] for more about background of (1.12).

Due to the well-known formula $\dim \mathfrak{g} = r(h + 1)$, the right hand side of (1.12) is equal to the number

$$(1.13) \qquad \frac{r(\ell h - h^\vee)}{h^\vee + \ell}.$$

In fact, we prove a functional generalization of Theorem 1.9, following the ideas of [GT, FS, C, Nkn].

Theorem 1.10 (Functional dilogarithm identities). *Suppose that a family of positive real numbers $\{Y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}_\ell\}$ satisfies $\mathbb{Y}_\ell(B_r)$. Then, we have the identities*

$$(1.14) \qquad \frac{6}{\pi^2} \sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\ 0 \le u < 2(h^\vee + \ell)}} L\left(\frac{Y_m^{(a)}(u)}{1 + Y_m^{(a)}(u)}\right) = 4r(\ell h - h^\vee) = 4r(2r\ell - 2r + 1),$$

$$(1.15) \qquad \frac{6}{\pi^2} \sum_{\substack{(a,m,u) \in \mathcal{I}_\ell \\ 0 \le u < 2(h^\vee + \ell)}} L\left(\frac{1}{1 + Y_m^{(a)}(u)}\right) = 4\ell(r\ell + \ell - 1).$$

The two identities (1.14) and (1.15) are equivalent to each other due to (1.9), since the sum of the right hand sides is equal to $4(h^\vee + \ell)(r\ell + \ell - r)$, which is the total number of $(a, m, u) \in I_\ell$ with $0 \le u < 2(h^\vee + \ell)$.

It is clear that Theorem 1.9 follows form Theorem 1.10 by considering a *constant solution* $Y_m^{(a)} = Y_m^{(a)}(u)$ of $\mathbb{Y}_\ell(B_r)$ in the variable u.

1.5. **Outline of method and contents.** Let us briefly explain the idea of our proof of the main results, Theorems 1.5, 1.6, and 1.10.

To start up, we identify the T and Y-systems, $\mathbb{T}_\ell(B_r)$ and $\mathbb{Y}_\ell(B_r)$ in Definitions 1.1 and 1.2 as systems of relations for cluster variables and coefficients of a cluster algebra, respectively. This procedure is mostly parallel to the simply laced case [FZ2, FZ3, Ke1, DiK, HL, IIKNS, KNS3], but necessarily more complicated. For example, unlike the simply laced ones, the composite mutation which generates the translation of the spectral parameter u is neither of *bipartite type* nor obviously related to the *Coxeter element* of a certain Weyl group. This is not a serious problem, though. A real problem is that the arising quiver $Q_\ell(B_r)$ for the cluster

algebra, which is seen in Figure 1 in Section 2.3, is not a familiar one in the representation theory of quivers; In particular, we have no known or obvious periodicity result. This is the main obstacle to a straightforward application of the method in [Ke1, IIKNS, Ke2], where the periodicities of the T and Y-systems in the simply laced case were derived from the periodicity in the corresponding cluster category.

The key to bypass this obstacle is to consider the *tropical Y-system.* The tropical Y-system is the tropicalization of the Y-system, or more generally, the exchange relations of *the coefficients in the tropical semifield* (called *principal coefficients* in [FZ3]) of a given cluster algebra. In fact, it was already used by Fomin-Zelevinsky [FZ2] as a main tool in the proof of the periodicity of the Y-systems in the simply laced case at level 2. In addition, we make two crucial observations.

Observation 1. *The periodicities of cluster variables and coefficients follow from the periodicity of principal coefficients.*

See Theorem 5.1 for a precise statement. This claim was essentially conjectured by Fomin-Zelevinsky [FZ3, Conjecture 4.7]. We prove the claim for the cluster algebra associated with any *skew-symmetric matrix*, or equivalently, with any *quiver.* To prove it, we use the recent result by Plamondon [Pl1, Pl2] on the categorification of the cluster algebra associated with an *arbitrary* quiver. It is so far the most general formulation of the categorification by *2-Calabi-Yau categories* recently developed by various authors in particular cases (see [A] and the references therein). Since each principal coefficient tuple carries the complete information of the corresponding object in the category through *index*, the periodicity of principal coefficients implies the same periodicity of objects in the category. Therefore, it also implies the same periodicities of cluster variables and coefficients by categorification.

Observation 2. *The tropical Y-system for $\mathbb{Y}_\ell(B_r)$ has a remarkable 'factorization property' so that its periodicity can be directly verified.*

Such a factorization property of the tropical Y-system was first noticed by [Nkn] in the proof of the dilogarithm identities in the simply laced case. It roughly means that the tropical Y-system at a higher level splits into the level 2 pieces and the type A pieces. Moreover, each piece can be described in terms of the piecewise-linear analogue of the simple reflections of a certain Weyl group introduced by [FZ2]. Therefore, the periodicity is tractable.

Combining these two observations, we obtain the desired periodicities in Theorems 1.5 and 1.6.

The tropical Y-system plays a central role not only in the periodicity but also in the dilogarithm identity. The following observation was made in [Nkn] in the simply laced case.

Observation 3. *The dilogarithm identity reduces to the positivity/negativity property of the tropical Y-system for $\mathbb{Y}_\ell(B_r)$.*

Shortly speaking, *"the tropical Y-system knows everything"* is our slogan.

The organization of the paper is as follows. In Section 2, we introduce a quiver $Q_\ell(B_r)$ and identify the T and Y-systems with systems of relations for cluster variables and coefficients of the cluster algebra associated with $Q_\ell(B_r)$ (Theorems 2.7 and 2.10). In Section 3, we study the tropical Y-system at level 2 and derive the periodicity and the positivity/negativity property (Proposition 3.2). In Section 4, we study the tropical Y-system at higher levels and show the factorization property. As a result, we obtain the periodicity (Theorems 4.2) and the positivity/negativity property (Theorem 4.3). In Section 5, based on the result by Plamondon, we

present a general theorem stating Observation 1 above (Theorem 5.1) for the cluster algebra associated with any skew-symmetric matrix. As corollaries, we obtain the periodicities of T and Y-systems in Theorem 1.5 and 1.6. In Section 6, using Theorem 4.3, we prove the dilogarithm identity in Theorem 1.10. In Section 7, as a feedback of the newly introduced method, we give an alternative and simplified proof of the periodicities of the T and Y-systems associated with pairs of simply laced Dynkin diagrams, which were formerly proved by [Ke1, IIKNS, Ke2].

2. CLUSTER ALGEBRAIC FORMULATION

The systems $\mathbb{T}_\ell(B_r)$ and $\mathbb{Y}_\ell(B_r)$ are naturally identified with systems of relations for cluster variables and coefficients, respectively, of a certain cluster algebra. They are mostly parallel to the simply laced case [FZ2, FZ3, Ke1, DiK, HL, IIKNS, KNS3], but necessarily more complicated.

2.1. **Groceries for cluster algebras.** Here we collect basic definitions for cluster algebras to fix the convention and notation, mainly following [FZ3]. For further necessary definitions and information for cluster algebras, see [FZ3].

Let I be a finite index set throughout this subsection.

(i) Semifield. A *semifield* $(\mathbb{P}, \oplus, \cdot)$ is an abelian multiplicative group endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The following three examples are relevant in this paper.

(a) *Trivial semifield.* The *trivial semifield* $\mathbf{1} = \{1\}$ consists of the multiplicative identity element 1 with $1 \oplus 1 = 1$.

(b) *Universal semifield.* For an I-tuple of variables $y = (y_i)_{i \in I}$, the *universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y)$ consists of all the rational functions of the form $P(y)/Q(y)$ (subtraction-free rational expressions), where $P(y)$ and $Q(y)$ are the nonzero polynomials in y_i's with *nonnegative* integer coefficients. The multiplication and the addition are given by the usual ones of rational functions.

(c) *Tropical semifield.* For an I-tuple of variables $y = (y_i)_{i \in I}$, the *tropical semifield* $\mathrm{Trop}(y)$ is the abelian multiplicative group freely generated by the variables y_i's endowed with the addition $\oplus$

$$(2.1) \qquad \prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}.$$

(ii) Mutations of matrix and quiver. An integer matrix $B = (B_{ij})_{i,j \in I}$ is *skew-symmetrizable* if there is a diagonal matrix $D = \mathrm{diag}(d_i)_{i \in I}$ with $d_i \in \mathbb{N}$ such that DB is skew-symmetric. For a skew-symmetrizable matrix B and $k \in I$, another matrix $B' = \mu_k(B)$, called the *mutation of B at k*, is defined by

$$(2.2) \qquad B'_{ij} = \begin{cases} -B_{ij} & i = k \text{ or } j = k, \\ B_{ij} + \frac{1}{2}(|B_{ik}|B_{kj} + B_{ik}|B_{kj}|) & \text{otherwise.} \end{cases}$$

The matrix $\mu_k(B)$ is also skew-symmetrizable.

It is standard to represent a *skew-symmetric* (integer) matrix $B = (B_{ij})_{i,j \in I}$ by a *quiver Q without loops or 2-cycles*. The set of the vertices of Q is given by I, and we put B_{ij} arrows from i to j if $B_{ij} > 0$. The mutation $Q' = \mu_k(Q)$ of quiver Q is given by the following rule: For each pair of an incoming arrow $i \to k$ and an

outgoing arrow $k \to j$ in Q, add a new arrow $i \to j$. Then, remove a maximal set of pairwise disjoint 2-cycles. Finally, reverse all arrows incident with k.

(iii) Exchange relation of coefficient tuple. Let $\mathbb{P}$ be a given semifield. For an I-tuple $y = (y_i)_{i \in I}$, $y_i \in \mathbb{P}$ and $k \in I$, another I-tuple y' is defined by the *exchange relation*

$$(2.3) \qquad y_i' = \begin{cases} y_k^{-1} & i = k, \\ y_i \left(\dfrac{y_k}{1 \oplus y_k} \right)^{B_{ki}} & i \neq k, \; B_{ki} \geq 0, \\ y_i (1 \oplus y_k)^{-B_{ki}} & i \neq k, \; B_{ki} \leq 0. \end{cases}$$

(iv) Exchange relation of cluster. Let $\mathbb{QP}$ be the quotient field of the group ring $\mathbb{ZP}$ of $\mathbb{P}$, and let $\mathbb{QP}(z)$ be the rational function field of algebraically independent variables, say, $z = (z_i)_{i \in I}$ over $\mathbb{QP}$. For an I-tuple $x = (x_i)_{i \in I}$ which is a free generating set of $\mathbb{QP}(z)$ and $k \in I$, another I-tuple x' is defined by the *exchange relation*

$$(2.4) \qquad x_i' = \begin{cases} x_i & i \neq k, \\ \dfrac{y_k \prod_{j:B_{jk}>0} x_j^{B_{jk}} + \prod_{j:B_{jk}<0} x_j^{-B_{jk}}}{(1 \oplus y_k) x_k} & i = k. \end{cases}$$

(v) Seed mutation. For the above triplet (B, x, y) in (ii)–(iv), which is called a *seed*, the mutation $\mu_k(B, x, y) = (B', x', y')$ at k is defined by combining (2.2), (2.3), and (2.4).

(vi) Cluster algebra. Fix a semifield $\mathbb{P}$ and a seed (*initial seed*) (B, x, y), where $x = (x_i)_{i \in I}$ are algebraically independent variables over $\mathbb{QP}$. Starting from (B, x, y), iterate mutations and collect all the seeds (B', x', y'). We call y' and y_i' a *coefficient tuple* and a *coefficient*, respectively. We call x' and $x_i' \in \mathbb{QP}(x)$, a *cluster* and a *cluster variable*, respectively. The *cluster algebra* $\mathcal{A}(B, x, y)$ *with coefficients in* $\mathbb{P}$ is the $\mathbb{ZP}$-subalgebra of the rational function field $\mathbb{QP}(x)$ generated by all the cluster variables.

(vii) Cluster pattern. Let $I = \{1, \ldots, n\}$, and let $\mathbb{T}_n$ be the n-regular tree whose edges are labeled by the numbers $1, \ldots, n$. A *cluster pattern* is the assignment of a seed $(B(t), x(t), y(t))$ for each vertex $t \in \mathbb{T}_n$ such that the seeds assigned to the endpoints of any edge $t \overset{k}{\text{---}} t'$ are obtained from each other by the seed mutation at k. Take $t_0 \in \mathbb{T}_n$ arbitrarily, and consider the cluster algebra $\mathcal{A}(B(t_0), x(t_0), y(t_0))$. Then, $x(t)$ and $y(t)$ ($t \in \mathbb{T}_n$) are a cluster and a coefficient tuple of $\mathcal{A}(B(t_0), x(t_0), y(t_0))$.

(viii) F-polynomial. The cluster algebra $\mathcal{A}(B, x, y)$ with coefficients in the tropical semifield $\mathrm{Trop}(y)$ is called the cluster algebra with *principal coefficients*. There, each cluster variable x_i' is an element in $\mathbb{Z}[x^{\pm 1}, y]$. The F-polynomial $F_i'(y) \in \mathbb{Z}[y]$ (for x_i') is defined as the specialization of x_i' with $x_i = 1$ ($i \in I$).

2.2. Parity decompositions of T and Y-systems.
For a triplet $(a, m, u) \in \mathcal{I}_\ell$, we set the 'parity conditions' $\mathbf{P}_+$ and $\mathbf{P}_-$ by

$$(2.5) \qquad \mathbf{P}_+ : \; 2u \text{ is even if } a \neq r; \; m + 2u \text{ is even if } a = r,$$

$$(2.6) \qquad \mathbf{P}_- : \; 2u \text{ is odd if } a \neq r; \; m + 2u \text{ is odd if } a = r.$$

We write, for example, $(a, m, u) : \mathbf{P}_+$ if (a, m, u) satisfies $\mathbf{P}_+$. We have $\mathcal{I}_\ell = \mathcal{I}_{\ell+} \sqcup \mathcal{I}_{\ell-}$, where $\mathcal{I}_{\ell\varepsilon}$ is the set of all $(a, m, u) : \mathbf{P}_\varepsilon$.

Define $\mathcal{T}_\ell^\circ(B_r)_\varepsilon$ ($\varepsilon = \pm$) to be the subring of $\mathcal{T}_\ell^\circ(B_r)$ generated by $T_m^{(a)}(u)$ ($(a,m,u) \in \mathcal{I}_{\ell\varepsilon}$). Then, we have $\mathcal{T}_\ell^\circ(B_r)_+ \simeq \mathcal{T}_\ell^\circ(B_r)_-$ by $T_m^{(a)}(u) \mapsto T_m^{(a)}(u + \frac{1}{2})$ and

$$(2.7) \qquad \mathcal{T}_\ell^\circ(B_r) \simeq \mathcal{T}_\ell^\circ(B_r)_+ \otimes_{\mathbb{Z}} \mathcal{T}_\ell^\circ(B_r)_-.$$

For a triplet $(a,m,u) \in \mathcal{I}_\ell$, we set another 'parity conditions' $\mathbf{P}'_+$ and $\mathbf{P}'_-$ by

$$(2.8) \qquad \mathbf{P}'_+ : \ 2u \text{ is even if } a \neq r;\ m + 2u \text{ is odd if } a = r,$$

$$(2.9) \qquad \mathbf{P}'_- : \ 2u \text{ is odd if } a \neq r;\ m + 2u \text{ is even if } a = r.$$

We have $\mathcal{I}_\ell = \mathcal{I}'_{\ell+} \sqcup \mathcal{I}'_{\ell-}$, where $\mathcal{I}'_{\ell\varepsilon}$ is the set of all $(a,m,u) : \mathbf{P}'_\varepsilon$. We also have

$$(2.10) \qquad (a,m,u) : \mathbf{P}'_+ \iff (a,m,u \pm \tfrac{1}{t_a}) : \mathbf{P}_+.$$

Define $\mathcal{Y}_\ell^\circ(B_r)_\varepsilon$ ($\varepsilon = \pm$) to be the subgroup of $\mathcal{Y}_\ell^\circ(B_r)$ generated by $Y_m^{(a)}(u)$, $1 + Y_m^{(a)}(u)$ ($(a,m,u) \in \mathcal{I}'_{\ell\varepsilon}$). Then, we have $\mathcal{Y}_\ell^\circ(B_r)_+ \simeq \mathcal{Y}_\ell^\circ(B_r)_-$ by $Y_m^{(a)}(u) \mapsto Y_m^{(a)}(u + \frac{1}{2})$, $1 + Y_m^{(a)}(u) \mapsto 1 + Y_m^{(a)}(u + \frac{1}{2})$, and

$$(2.11) \qquad \mathcal{Y}_\ell^\circ(B_r) \simeq \mathcal{Y}_\ell^\circ(B_r)_+ \times \mathcal{Y}_\ell^\circ(B_r)_-.$$

2.3. **Quiver $Q_\ell(B_r)$.** With type B_r and $\ell \geq 2$ we associate the quiver $Q_\ell(B_r)$ by Figure 1, where, in addition, we assign the empty or filled circle $\circ/\bullet$ and the sign $+/-$ to each vertex. Let $B_\ell(B_r)$ be the skew-symmetric matrix corresponding to $Q_\ell(B_r)$ as defined in Section 2.1 (ii). (Unfortunately, there is an obvious confliction between two standard notations using symbol B. We hope it does not cause serious confusion to the reader.)

Let us choose the index set $\mathbf{I}$ of the vertices of $Q_\ell(B_r)$ so that $\mathbf{i} = (i, i') \in \mathbf{I}$ represents the vertex at the i'th row (from the bottom) of the ith column (from the left). Thus, $i = 1, \ldots, 2r - 1$, and $i' = 1, \ldots, \ell - 1$ if $i \neq r$ and $i' = 1, \ldots, 2\ell - 1$ if $i = r$. We use a natural notation $\mathbf{I}^\circ$ (resp. $\mathbf{I}_+^\circ$) for the set of the vertices $\mathbf{i}$ with property $\circ$ (resp. $\circ$ and $+$), and so on. We have $\mathbf{I} = \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet = \mathbf{I}_+^\circ \sqcup \mathbf{I}_-^\circ \sqcup \mathbf{I}_+^\bullet \sqcup \mathbf{I}_-^\bullet$.

We define composite mutations,

$$(2.12) \qquad \mu_+^\circ = \prod_{\mathbf{i} \in \mathbf{I}_+^\circ} \mu_{\mathbf{i}}, \quad \mu_-^\circ = \prod_{\mathbf{i} \in \mathbf{I}_-^\circ} \mu_{\mathbf{i}}, \quad \mu_+^\bullet = \prod_{\mathbf{i} \in \mathbf{I}_+^\bullet} \mu_{\mathbf{i}}, \quad \mu_-^\bullet = \prod_{\mathbf{i} \in \mathbf{I}_-^\bullet} \mu_{\mathbf{i}}.$$

Note that they do not depend on the order of the product.

Let $\boldsymbol{r}$ be the involution acting on $\mathbf{I}$ by the left-right reflection. Let $\boldsymbol{\omega}$ be the involution acting on $\mathbf{I}$ by the 180° rotation. Let $\boldsymbol{r}(Q_\ell(B_r))$ and $\boldsymbol{\omega}(Q_\ell(B_r))$ denote the quivers induced from $Q_\ell(B_r)$ by $\boldsymbol{r}$ and $\boldsymbol{\omega}$, respectively. For example, if there is an arrow $\mathbf{i} \to \mathbf{j}$ in $Q_\ell(B_r)$, then, there is an arrow $\boldsymbol{r}(\mathbf{i}) \to \boldsymbol{r}(\mathbf{j})$ in $\boldsymbol{r}(Q_\ell(B_r))$. For a quiver Q, Q^{op} denotes the opposite quiver.

Lemma 2.1. *Let $Q = Q_\ell(B_r)$.*
(i) We have a periodic sequence of mutations of quivers

$$(2.13) \qquad Q \overset{\mu_+^\bullet \mu_+^\circ}{\longleftrightarrow} Q^{\mathrm{op}} \overset{\mu_-^\bullet}{\longleftrightarrow} \boldsymbol{r}(Q) \overset{\mu_+^\bullet \mu_-^\circ}{\longleftrightarrow} \boldsymbol{r}(Q)^{\mathrm{op}} \overset{\mu_-^\bullet}{\longleftrightarrow} Q.$$

(ii) $\boldsymbol{\omega}(Q) = Q$ if $h^\vee + \ell$ is even, and $\boldsymbol{\omega}(Q) = \boldsymbol{r}(Q)$ if $h^\vee + \ell$ is odd.
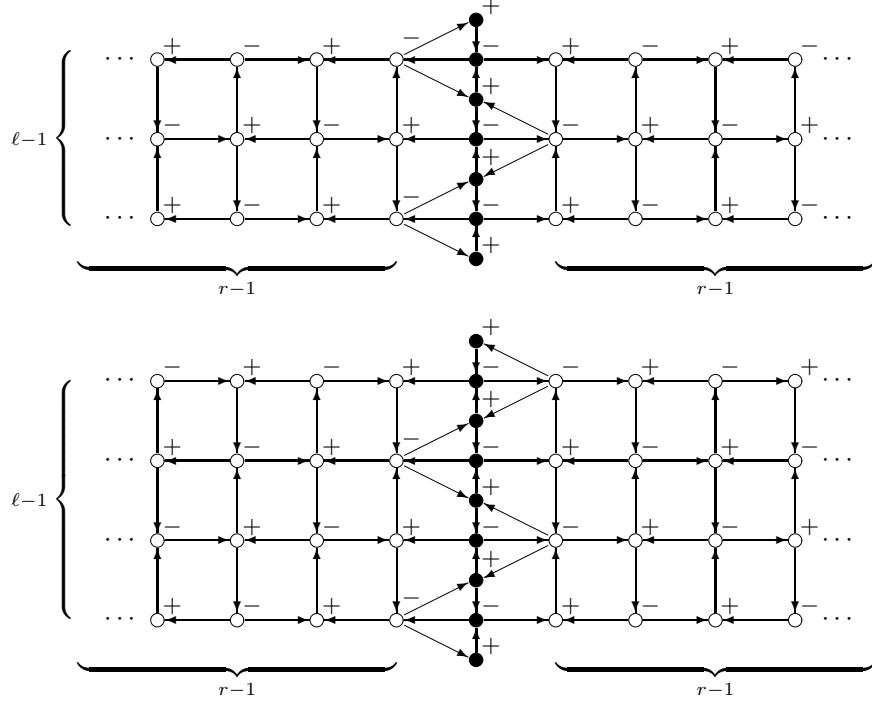
FIGURE 1. The quiver $Q_\ell(B_r)$ for even ℓ (upper) and for odd ℓ (lower).

Example 2.2. The sequence (2.13) for $Q = Q_2(B_3)$ is given below.
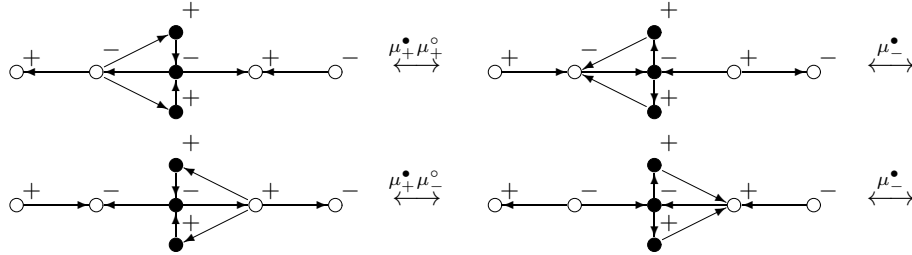
2.4. **Cluster algebra $\mathcal{A}(B, x, y)$ and coefficient group $\mathcal{G}(B, y)$.** For the matrix $B = (B_{\mathbf{ij}})_{\mathbf{i,j} \in \mathbf{I}} = B_\ell(B_r)$, let $\mathcal{A}(B, x, y)$ be the *cluster algebra with coefficients in the universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y)$, where (B, x, y) is the initial seed. (Here we use the symbol $+$ instead of $\oplus$ in $\mathbb{Q}_{\mathrm{sf}}(y)$, since it is the ordinary addition of subtraction-free expressions of rational functions of y.)

To our purpose, it is natural to introduce not only the 'ring of cluster variables' but also the 'group of coefficients'.

Definition 2.3. The *coefficient group $\mathcal{G}(B, y)$ associated with* $\mathcal{A}(B, x, y)$ is the multiplicative subgroup of the semifield $\mathbb{Q}_{\mathrm{sf}}(y)$ generated by all the coefficients $y'_{\mathbf{i}}$ of $\mathcal{A}(B, x, y)$ together with $1 + y'_{\mathbf{i}}$.

In view of Lemma 2.1 we set $x(0) = x$, $y(0) = y$ and define clusters $x(u) = (x_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ $(u \in \frac{1}{2}\mathbb{Z})$ and coefficient tuples $y(u) = (y_{\mathbf{i}}(u))_{\mathbf{i} \in \mathbf{I}}$ $(u \in \frac{1}{2}\mathbb{Z})$ by the sequence of mutations

$$(2.14) \qquad \begin{aligned} &\cdots \xleftrightarrow{\mu_{-}^{\bullet}} (B, x(0), y(0)) \xleftrightarrow{\mu_{+}^{\bullet}\mu_{+}^{\circ}} (-B, x(\tfrac{1}{2}), y(\tfrac{1}{2})) \\ &\xleftrightarrow{\mu_{-}^{\bullet}} (\boldsymbol{r}(B), x(1), y(1)) \xleftrightarrow{\mu_{+}^{\bullet}\mu_{+}^{\circ}} (-\boldsymbol{r}(B), x(\tfrac{3}{2}), y(\tfrac{3}{2})) \xleftrightarrow{\mu_{-}^{\bullet}} \cdots, \end{aligned}$$

where $\boldsymbol{r}(B) = B'$ is defined by $B'_{\boldsymbol{r}(\mathbf{i})\boldsymbol{r}(\mathbf{j})} = B_{\mathbf{ij}}$.

For a pair $(\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z}$, we set the parity condition $\mathbf{p}_+$ and $\mathbf{p}_-$ by

$$(2.15) \qquad \mathbf{p}_+ : \begin{cases} \mathbf{i} \in \mathbf{I}_+^{\circ} \sqcup \mathbf{I}_+^{\bullet} & u \equiv 0 \\ \mathbf{i} \in \mathbf{I}_-^{\bullet} & u \equiv \tfrac{1}{2}, \tfrac{3}{2} \\ \mathbf{i} \in \mathbf{I}_-^{\circ} \sqcup \mathbf{I}_+^{\bullet} & u \equiv 1, \end{cases} \qquad \mathbf{p}_- : \begin{cases} \mathbf{i} \in \mathbf{I}_+^{\circ} \sqcup \mathbf{I}_+^{\bullet} & u \equiv \tfrac{1}{2} \\ \mathbf{i} \in \mathbf{I}_-^{\bullet} & u \equiv 0, 1 \\ \mathbf{i} \in \mathbf{I}_-^{\circ} \sqcup \mathbf{I}_+^{\bullet} & u \equiv \tfrac{3}{2}, \end{cases}$$

where $\equiv$ is modulo $2\mathbb{Z}$. We have

$$(2.16) \qquad (\mathbf{i}, u) : \mathbf{p}_+ \quad \Longleftrightarrow \quad (\mathbf{i}, u + \tfrac{1}{2}) : \mathbf{p}_-.$$

Each $(\mathbf{i}, u) : \mathbf{p}_+$ is a mutation point of (2.14) in the forward direction of u, and each $(\mathbf{i}, u) : \mathbf{p}_-$ is so in the backward direction of u. Notice that there are also some $(\mathbf{i}, u)$ which do not satisfy $\mathbf{p}_+$ nor $\mathbf{p}_-$, and are not mutation points of (2.14); explicitly, they are $(\mathbf{i}, u)$ with $\mathbf{i} \in \mathbf{I}_+^{\circ}$, $u \equiv 1, \tfrac{3}{2}$ mod $2\mathbb{Z}$, or with $\mathbf{i} \in \mathbf{I}_-^{\circ}$, $u \equiv 0, \tfrac{1}{2}$ mod $2\mathbb{Z}$. Consequently, we have the following relations for $(\mathbf{i}, u) : \mathbf{p}_\pm$.

$$(2.17) \qquad x_{\mathbf{i}}(u) = x_{\mathbf{i}}(u \mp \tfrac{1}{2}) \quad (\mathbf{i} \in \mathbf{I}^{\bullet}),$$

$$(2.18) \qquad x_{\mathbf{i}}(u) = x_{\mathbf{i}}(u \mp \tfrac{1}{2}) = x_{\mathbf{i}}(u \mp 1) = x_{\mathbf{i}}(u \mp \tfrac{3}{2}), \quad (\mathbf{i} \in \mathbf{I}^{\circ}),$$

$$(2.19) \qquad y_{\mathbf{i}}(u) = y_{\mathbf{i}}(u \pm \tfrac{1}{2})^{-1}.$$

There is a correspondence between the parity condition $\mathbf{p}_+$ here and $\mathbf{P}_+$, $\mathbf{P}'_+$ in (2.5) and (2.8).

Lemma 2.4. *Below $\equiv$ means the equivalence modulo $2\mathbb{Z}$.*
(i) The map

$$(2.20) \qquad \begin{aligned} g : \quad &\mathcal{I}_{\ell+} \quad \rightarrow \quad \{(\mathbf{i}, u) : \mathbf{p}_+\} \\ &(a, m, u - \tfrac{1}{t_a}) \quad \mapsto \quad \begin{cases} ((a, m), u) & a \neq r; \ r + a + m + u \equiv 1 \\ ((2r - a, m), u) & a \neq r; \ r + a + m + u \equiv 0 \\ ((r, m), u) & a = r \end{cases} \end{aligned}$$

is a bijection.
(ii) The map

$$(2.21) \qquad \begin{aligned} g' : \quad &\mathcal{I}'_{\ell+} \quad \rightarrow \quad \{(\mathbf{i}, u) : \mathbf{p}_+\} \\ &(a, m, u) \quad \mapsto \quad \begin{cases} ((a, m), u) & a \neq r; \ r + a + m + u \equiv 1 \\ ((2r - a, m), u) & a \neq r; \ r + a + m + u \equiv 0 \\ ((r, m), u) & a = r \end{cases} \end{aligned}$$

is a bijection.

Proof. They can be easily confirmed by looking at the example in Figures 2–5. $\square$
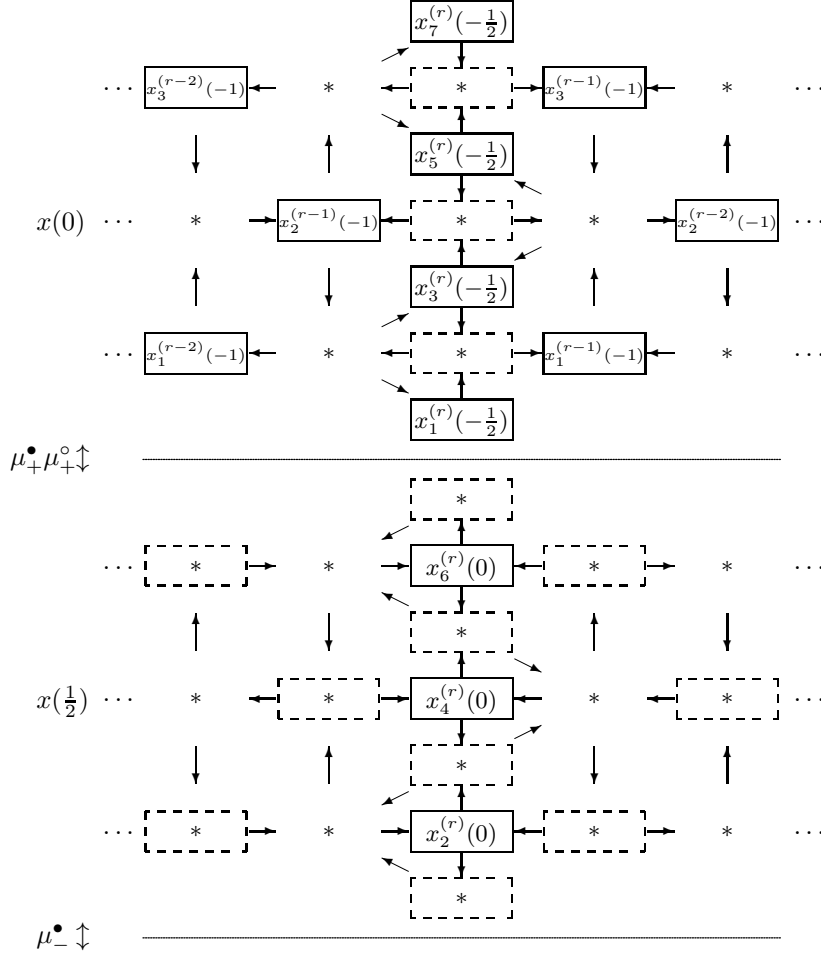
FIGURE 2. (Continues to Figure 3) Label of cluster variables $x_{\mathbf{i}}(u)$ by $\mathcal{I}_{\ell+}$ for B_r, $\ell = 4$. The variables framed by solid/dashed lines satisfy the condition $\mathbf{p}_+/\mathbf{p}_-$, respectively.

We introduce alternative labels $x_{\mathbf{i}}(u) = x_m^{(a)}(u - 1/t_a)$ $((a, m, u - 1/t_a) \in \mathcal{I}_{\ell+})$ for $(\mathbf{i}, u) = g((a, m, u - 1/t_a))$ and $y_{\mathbf{i}}(u) = y_m^{(a)}(u)$ $((a, m, u) \in \mathcal{I}'_{\ell+})$ for $(\mathbf{i}, u) = g'((a, m, u))$, respectively, as in Figures 2–5. They will be used below (and also in Section 6) to relate the T and Y-systems with $\mathcal{A}(B, x, y)$ and $\mathcal{G}(B, y)$.

2.5. **T-system and cluster algebra.** The T-system $\mathbb{T}_\ell(B_r)$ naturally appears as a system of relations among the cluster variables $x_{\mathbf{i}}(u)$ in the trivial evaluation of coefficients. (The quiver $Q_\ell(B_r)$ is designed to do so.) Let $\mathcal{A}(B, x)$ be the cluster algebra with trivial coefficients, where (B, x) is the initial seed. Let $\mathbf{1} = \{1\}$ be the *trivial semifield* and $\pi_{\mathbf{1}} : \mathbb{Q}_{\mathrm{sf}}(y) \to \mathbf{1}$, $y_{\mathbf{i}} \mapsto 1$ be the projection. Let $[x_{\mathbf{i}}(u)]_{\mathbf{1}}$ denote the image of $x_{\mathbf{i}}(u)$ by the algebra homomorphism $\mathcal{A}(B, x, y) \to \mathcal{A}(B, x)$ induced from $\pi_{\mathbf{1}}$. It is called the *trivial evaluation*.
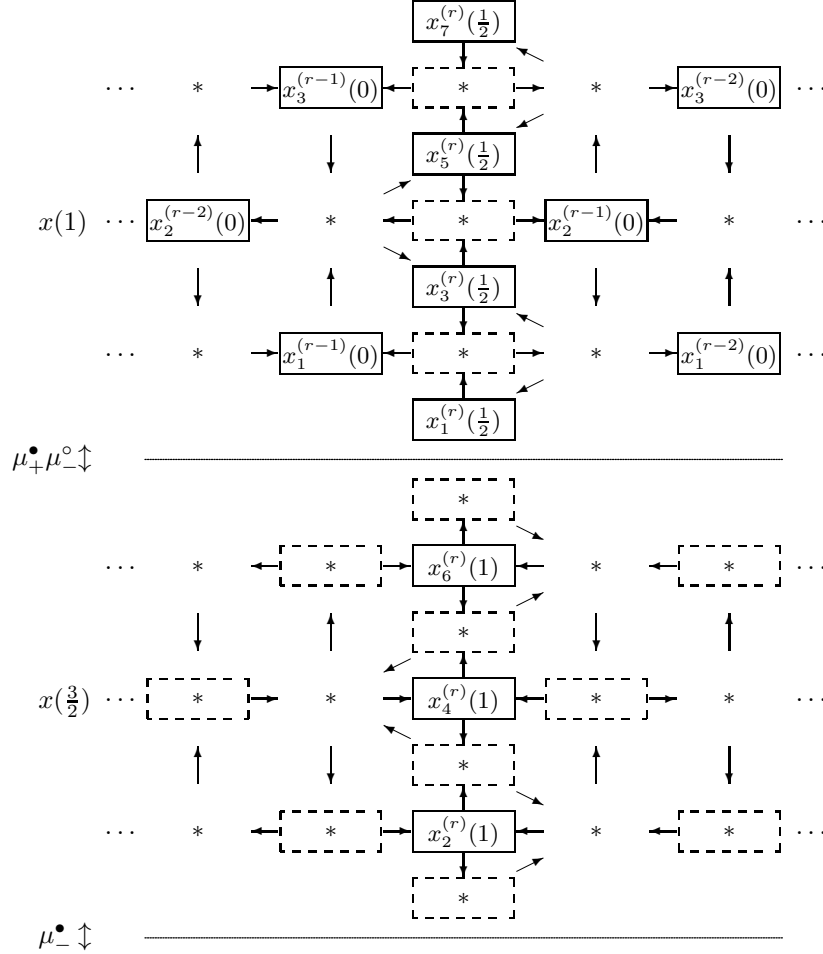
Recall that $G(b, k, v; a, m, u)$ is defined in (1.5).

$$x_7^{(r)}(\tfrac{1}{2})$$

$\cdots$ $*$ $\to x_3^{(r-1)}(0) \leftarrow$ $*$ $\to$ $*$ $\to x_3^{(r-2)}(0)$ $\cdots$

$$x_5^{(r)}(\tfrac{1}{2})$$

$x(1)$ $\cdots$ $x_2^{(r-2)}(0) \leftarrow$ $*$ $\leftarrow$ $*$ $x_2^{(r-1)}(0) \leftarrow$ $*$ $\cdots$

$$x_3^{(r)}(\tfrac{1}{2})$$

$\cdots$ $*$ $\to x_1^{(r-1)}(0) \leftarrow$ $*$ $\to$ $*$ $\to x_1^{(r-2)}(0)$ $\cdots$

$$x_1^{(r)}(\tfrac{1}{2})$$

$\mu_+^\bullet \mu_-^\circ \updownarrow$ _____

$*$

$\cdots$ $*$ $\leftarrow$ $*$ $\to x_6^{(r)}(1) \leftarrow$ $*$ $\leftarrow$ $*$ $\cdots$

$*$

$x(\tfrac{3}{2})$ $\cdots$ $*$ $\to$ $*$ $\to x_4^{(r)}(1) \leftarrow$ $*$ $\to$ $*$ $\cdots$

$*$

$\cdots$ $*$ $\leftarrow$ $*$ $\to x_2^{(r)}(1) \leftarrow$ $*$ $\leftarrow$ $*$ $\cdots$

$*$

$\mu_-^\bullet \updownarrow$ _____

FIGURE 3. (Continues from Figure 2).

Lemma 2.5. *The family* $\{x_m^{(a)}(u) \mid (a,m,u) \in \mathcal{I}_{\ell+}\}$ *satisfies a system of relations*

(2.22)
$$x_m^{(a)}\left(u - \tfrac{1}{t_a}\right) x_m^{(a)}\left(u + \tfrac{1}{t_a}\right) = \frac{y_m^{(a)}(u)}{1 + y_m^{(a)}(u)} \prod_{(b,k,v) \in \mathcal{I}_{\ell+}} x_k^{(b)}(v)^{G(b,k,v;\,a,m,u)}$$
$$+ \frac{1}{1 + y_m^{(a)}(u)} x_{m-1}^{(a)}(u) x_{m+1}^{(a)}(u),$$

where $(a,m,u) \in \mathcal{I}'_{+\ell}$. *In particular, the family* $\{[x_m^{(a)}(u)]_{\mathbf{1}} \mid (a,m,u) \in \mathcal{I}_{\ell+}\}$ *satisfies the T-system* $\mathbb{T}_\ell(B_r)$ *in* $\mathcal{A}(B,x)$ *by replacing* $T_m^{(a)}(u)$ *with* $[x_m^{(a)}(u)]_{\mathbf{1}}$.

Proof. An easy way to prove it is to represent all the relevant cluster variables in Figures 2 and 3 by $x_m^{(a)}(u)$ $(a,m,u) \in \mathcal{I}_{\ell+}$, then to apply mutations at $(\mathbf{i}, u) : \mathbf{p}_+$ in the figures. For example, consider the mutation at $((r,3), 0) : \mathbf{p}_+$. Then, by the
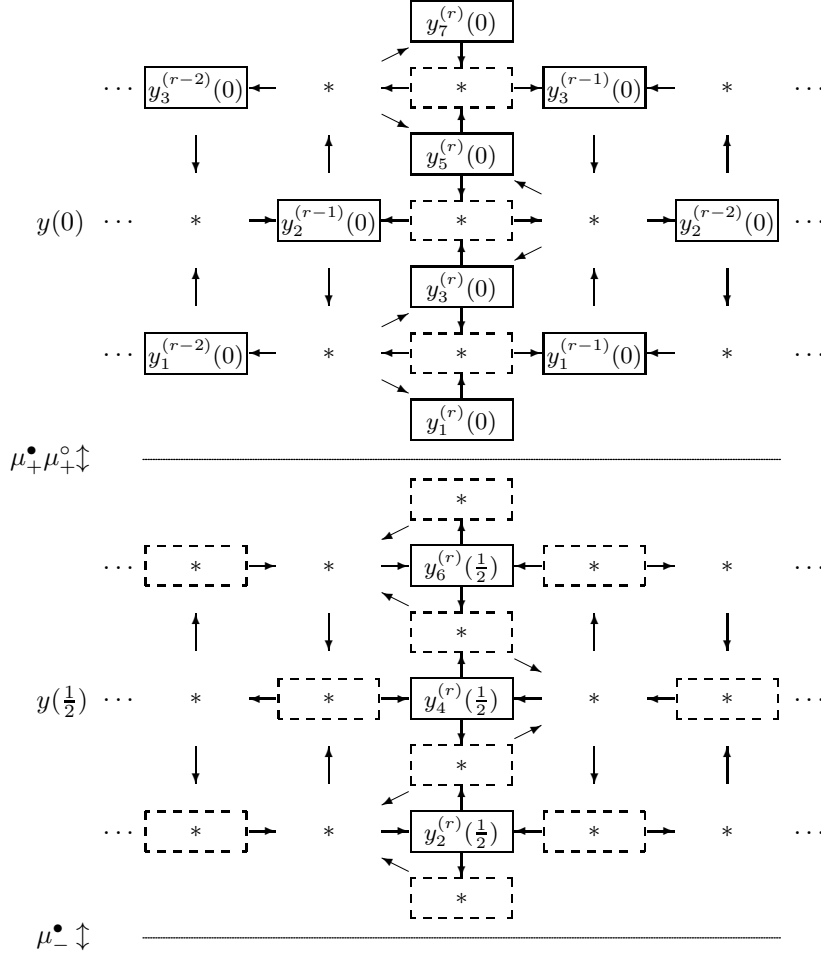
FIGURE 4. (Continues to Figure 5) Label of coefficients $y_{\mathbf{i}}(u)$ by $\mathcal{I}'_{\ell+}$ for B_r, $\ell = 4$. The variables framed by solid/dashed lines satisfy the condition $\mathbf{p}_+/\mathbf{p}_-$, respectively.

exchange relation, $x_3^{(r)}(-\frac{1}{2})$ is mutated to

$$(2.23) \quad \frac{1}{x_3^{(r)}\left(-\frac{1}{2}\right)} \left\{ \frac{y_3^{(r)}(0)}{1+y_3^{(r)}(0)} x_1^{(r-1)}(0) x_2^{(r-1)}(0) + \frac{1}{1+y_3^{(r)}(0)} x_2^{(r)}(0) x_4^{(r)}(0) \right\},$$

which should be equal to $x_3^{(r)}(\frac{1}{2})$. We note that Figures 2 and 3 are general enough for that purpose.   □

Definition 2.6. The *T-subalgebra* $\mathcal{A}_T(B,x)$ *of* $\mathcal{A}(B,x)$ *associated with the sequence* (2.14) *is the subalgebra of* $\mathcal{A}(B,x)$ *generated by* $[x_{\mathbf{i}}(u)]_{\mathbf{1}}$ $((\mathbf{i},u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z})$.

Theorem 2.7. *The ring* $\mathcal{T}_\ell^\circ(B_r)_+$ *is isomorphic to* $\mathcal{A}_T(B,x)$ *by the correspondence* $T_m^{(a)}(u) \mapsto [x_m^{(a)}(u)]_{\mathbf{1}}$.

$$y_7^{(r)}(1)$$

$$\cdots \quad * \quad \rightarrow y_3^{(r-1)}(1) \leftarrow \quad * \quad \rightarrow \quad * \quad \rightarrow y_3^{(r-2)}(1) \cdots$$

$$y_5^{(r)}(1)$$

$$y(1) \quad \cdots \quad y_2^{(r-2)}(1) \leftarrow \quad * \quad \leftarrow \quad * \quad \rightarrow y_2^{(r-1)}(1) \leftarrow \quad * \quad \cdots$$

$$y_3^{(r)}(1)$$

$$\cdots \quad * \quad \rightarrow y_1^{(r-1)}(1) \leftarrow \quad * \quad \rightarrow \quad * \quad \rightarrow y_1^{(r-2)}(1) \cdots$$

$$y_1^{(r)}(1)$$

$\mu_+^\bullet \mu_-^\circ \updownarrow$

$$\cdots \quad * \quad \leftarrow \quad * \quad \rightarrow y_6^{(r)}(\tfrac{3}{2}) \leftarrow \quad * \quad \leftarrow \quad * \quad \cdots$$

$$y(\tfrac{3}{2}) \quad \cdots \quad * \quad \rightarrow \quad * \quad \rightarrow y_4^{(r)}(\tfrac{3}{2}) \leftarrow \quad * \quad \rightarrow \quad * \quad \cdots$$

$$\cdots \quad * \quad \leftarrow \quad * \quad \rightarrow y_2^{(r)}(\tfrac{3}{2}) \leftarrow \quad * \quad \leftarrow \quad * \quad \cdots$$
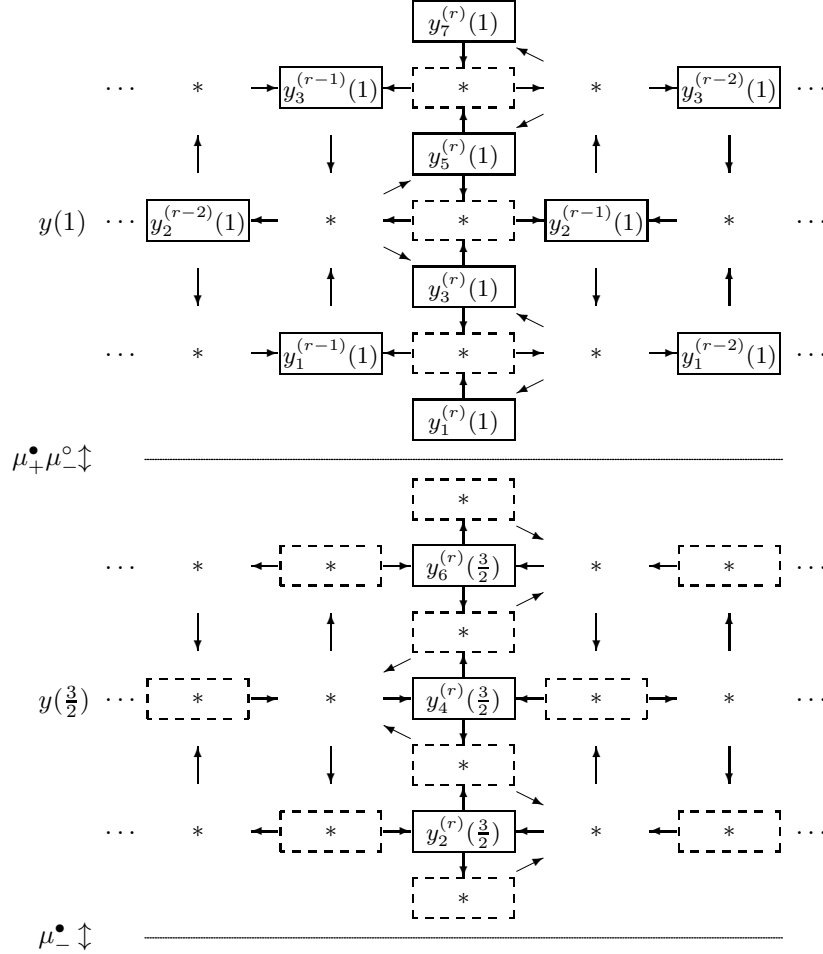
$\mu_-^\bullet \updownarrow$

FIGURE 5. (Continues from Figure 4).

Proof. First we note that $\mathcal{A}_T(B, x)$ is generated by $[x_\mathbf{i}(u)]_\mathbf{1}$ $((\mathbf{i}, u) : \mathbf{p}_+)$ by (2.17) and (2.18). Then, the claim follows from Lemma 2.5 in the same way as [IIKNS, Proposition 4.24]. □

2.6. **Y-system and cluster algebra.** The Y-system $\mathbb{Y}_\ell(B_r)$ also naturally appears as a system of relations among the coefficients $y_\mathbf{i}(u)$.

Lemma 2.8. *The family $\{y_m^{(a)}(u) \mid (a, m, u) \in \mathcal{I}'_{\ell+}\}$ satisfies the Y-system $\mathbb{Y}_\ell(B_r)$ by replacing $Y_m^{(a)}(u)$ with $y_m^{(a)}(u)$.*

Proof. This can be easily shown using Figures 4 and 5. For example, consider the mutation at $((r-1, 2), 0) : \mathbf{p}_+$. The coefficient $y_2^{(r-1)}(0)$ is mutated to $y_2^{(r-1)}(0)^{-1}$.

Then, at $u = \frac{1}{2}, 1, \frac{3}{2}$, the factors

$$
(2.24) \qquad 1 + y_4^{(r)}(\tfrac{1}{2}), \quad \frac{y_3^{(r-1)}(1)}{1 + y_1^{(r-1)}(1)}, \quad \frac{y_3^{(r-1)}(1)}{1 + y_1^{(r-1)}(1)},
$$
$$
1 + y_3^{(r)}(1), \quad 1 + y_5^{(r)}(1), \quad 1 + y_4^{(r)}(\tfrac{3}{2})
$$

are multiplied to $y_2^{(r-1)}(0)^{-1}$. The result should be equal to $y_2^{(r-1)}(2)$. $\qquad\square$

Definition 2.9. The *Y-subgroup* $\mathcal{G}_Y(B, y)$ *of* $\mathcal{G}(B, y)$ *associated with the sequence* (2.14) *is the subgroup of* $\mathcal{G}(B, y)$ *generated by* $y_{\mathbf{i}}(u)$ $((\mathbf{i}, u) \in \mathbf{I} \times \frac{1}{2}\mathbb{Z})$ *and* $1 + y_{\mathbf{i}}(u)$ $((\mathbf{i}, u) : \mathbf{p}_+ \text{ or } \mathbf{p}_-)$.

Notice that we excluded $1 + y_{\mathbf{i}}(u)$ for $(\mathbf{i}, u)$ not satisfying $\mathbf{p}_+$ nor $\mathbf{p}_-$. This is because such $(\mathbf{i}, u)$ is not a mutation point so that the factor $1 + y_{\mathbf{i}}(u)$ does not appear anywhere for the mutation sequence (2.14).

Theorem 2.10. *The group* $\mathcal{Y}_\ell^\circ(B_r)_+$ *is isomorphic to* $\mathcal{G}_Y(B, y)$ *by the correspondence* $Y_m^{(a)}(u) \mapsto y_m^{(a)}(u)$ *and* $1 + Y_m^{(a)}(u) \mapsto 1 + y_m^{(a)}(u)$.

Proof. We note that $\mathcal{G}_Y(B, y)$ is generated by $y_{\mathbf{i}}(u), 1 + y_{\mathbf{i}}(u)$ $((\mathbf{i}, u) : \mathbf{p}_+)$ by (2.19). Then, the claim follows from Lemma 2.8 in the same way as [KNS3, Theorem 6.19]. $\qquad\square$

3. Tropical Y-system at level 2

In this section we study the tropical version of the Y-system at level 2.

3.1. Tropical Y-system. Let $y = y(0)$ be the initial coefficient tuple of the cluster algebra $\mathcal{A}(B, x, y)$ with $B = B_\ell(B_r)$ in the previous section. Let $\mathrm{Trop}(y)$ be the *tropical semifield* for y. Let $\pi_{\mathbf{T}} : \mathbb{Q}_{\mathrm{sf}}(y) \to \mathrm{Trop}(y)$, $y_{\mathbf{i}} \mapsto y_{\mathbf{i}}$ be the projection. Let $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ and $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ denote the images of $y_{\mathbf{i}}(u)$ and $\mathcal{G}_Y(B, y)$ by the multiplicative group homomorphism induced from $\pi_{\mathbf{T}}$, respectively. They are called the *tropical evaluations*, and the resulting relations in the group $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ is called the *tropical Y-system*. They are first studied in [FZ2] for simply laced type at level 2 in our terminology.

We say a (Laurent) monomial $m = \prod_{\mathbf{i} \in \mathbf{I}} y_{\mathbf{i}}^{k_{\mathbf{i}}}$ is *positive* (resp. *negative*) if $m \neq 1$ and $k_{\mathbf{i}} \geq 0$ (resp. $k_{\mathbf{i}} \leq 0$) for any $\mathbf{i}$.

The next 'tropical mutation rule' for $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is general and useful.

Lemma 3.1. *Suppose that* y'' *is the coefficient tuple obtained from the mutation of another coefficient tuple* y' *at* $\mathbf{k}$ *with mutation matrix* B'. *Then, for any* $\mathbf{i} \neq \mathbf{k}$, *we have the rule:*

(i) $[y_{\mathbf{i}}'']_{\mathbf{T}} = [y_{\mathbf{i}}']_{\mathbf{T}}[y_{\mathbf{k}}']_{\mathbf{T}}$ *if one of the following conditions holds.*

 (a) $B_{\mathbf{ki}}' > 0$, *and* $[y_{\mathbf{k}}']_{\mathbf{T}}$ *is positive.*

 (b) $B_{\mathbf{ki}}' < 0$, *and* $[y_{\mathbf{k}}']_{\mathbf{T}}$ *is negative.*

(ii) $[y_{\mathbf{i}}'']_{\mathbf{T}} = [y_{\mathbf{i}}']_{\mathbf{T}}$ *if one of the following conditions holds.*

 (a) $B_{\mathbf{ki}}' = 0$.

 (b) $B_{\mathbf{ki}}' > 0$, *and* $[y_{\mathbf{k}}']_{\mathbf{T}}$ *is negative.*

 (c) $B_{\mathbf{ki}}' < 0$, *and* $[y_{\mathbf{k}}']_{\mathbf{T}}$ *is positive.*

Proof. This is an immediate consequence of the exchange relation (2.3) and (2.1). $\qquad\square$

The following properties of the tropical Y-system at level 2 will be the key in the entire method.

Proposition 3.2. *For $[\mathcal{G}_Y(B,y)]_\mathbf{T}$ with $B = B_2(B_r)$, the following facts hold.*

(i) Let u be in the region $0 \le u < 2$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_\mathbf{i}(u)]_\mathbf{T}$ is positive.

(ii) Let u be in the region $-h^\vee \le u < 0$.

  (a) Let $\mathbf{i} \in \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet_-$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_\mathbf{i}(u)]_\mathbf{T}$ is negative.

  (b) Let $\mathbf{i} \in \mathbf{I}^\bullet_+$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_\mathbf{i}(u)]_\mathbf{T}$ is negative for $u = -1, -3, \dots$ and positive for $u = -2, -4, \dots$.

(iii) $y_{ii'}(2) = y_{ii'}^{-1}$ if $i \ne r$ and $y_{r,4-i'}^{-1}$ if $i = r$.

(iv) $y_{ii'}(-h^\vee) = y_{2r-i,i'}^{-1}$.

Example 3.3. Consider the simplest case B_2. All the coefficients $[y_\mathbf{i}(u)]_\mathbf{T}$ in the region $-3 \le u \le 2$ are calculated with Lemma 3.1 and explicitly given in Figure 6. We continue to use the convention in Figures 2–5 such that the variables framed by solid/dashed lines satisfy the condition $\mathbf{p}_+/\mathbf{p}_-$. We recall that they are the mutation points for the forward/backward direction of u. In Figure 6, the configuration

$$
\begin{array}{ccc}
 & \cdot & \\
1 & 1 & \cdot \\
 & \cdot &
\end{array}
$$

for example represents the monomial $y_{11}y_{22}$, where $\cdot$ stands for 0. One can observe all the properties in Proposition 3.2 in Figure 6. Let us further observe that, in the region $-3 \le u < 0$, we have six negative monomials for $\mathbf{i} \in \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet_-$, $(\mathbf{i}, u) : \mathbf{p}_+$,

$$
\begin{array}{cccccc}
\cdot & \cdot & \text{-1} & \cdot & \cdot & \cdot \\
\cdot \;\; \cdot \;\; \text{-1} & \cdot \;\; \text{-1} \;\; \text{-1} & \text{-1} \;\; \text{-1} \;\; \text{-1} & \text{-1} \;\; \cdot \;\; \cdot & \cdot \;\; \text{-1} \;\; \cdot & \text{-1} \;\; \text{-1} \;\; \cdot \\
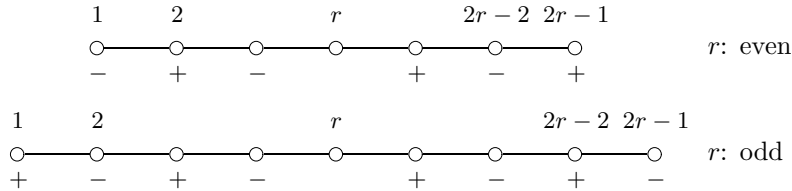\cdot & \cdot & \text{-1} & \cdot & \cdot & \cdot
\end{array}
$$

If we concentrate on the middle row, then they naturally correspond to the *positive roots* of type A_3

(3.1) $$\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3.$$

3.2. **Proof of Proposition 3.2.** For general r, one can directly verify (i) and (iii) as in Figure 6.

Note that (ii) and (iv) can be proved independently for each variable $y_\mathbf{i}$. (To be precise, we also need to assure that each monomial is not 1 in total. However, this can be easily followed up, so that we do not describe details here.) As for the powers of variables y_{r1} and y_{r3}, it is easy to verify the claim by direct calculations. Therefore, it is enough to prove (ii) and (iv) only for the powers of variables y_{i1} ($i \ne r$) and y_{r2}. To do that, we use the description of the tropical Y-system in the region $-h^\vee \le u < 0$ by the *root system of type A_{2r-1}*, following the spirit of [FZ2].

Let A_{2r-1} be the Dynkin diagram of type A with index set $J = \{1, \dots, 2r-1\}$. We assign the sign $+/-$ to vertices (except for r) of A_{2r-1} as inherited from $Q_2(B_r)$.

$$
\begin{array}{ccccccc}
1 & 2 & & r & & 2r-2 & 2r-1 \\
\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ \\
- & + & - & & + & - & +
\end{array}
\quad r\text{: even}
$$

$$
\begin{array}{cccccccc}
1 & 2 & & & r & & 2r-2 & 2r-1 \\
\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ\!\!-\!\!\!\!& \!\!\circ \\
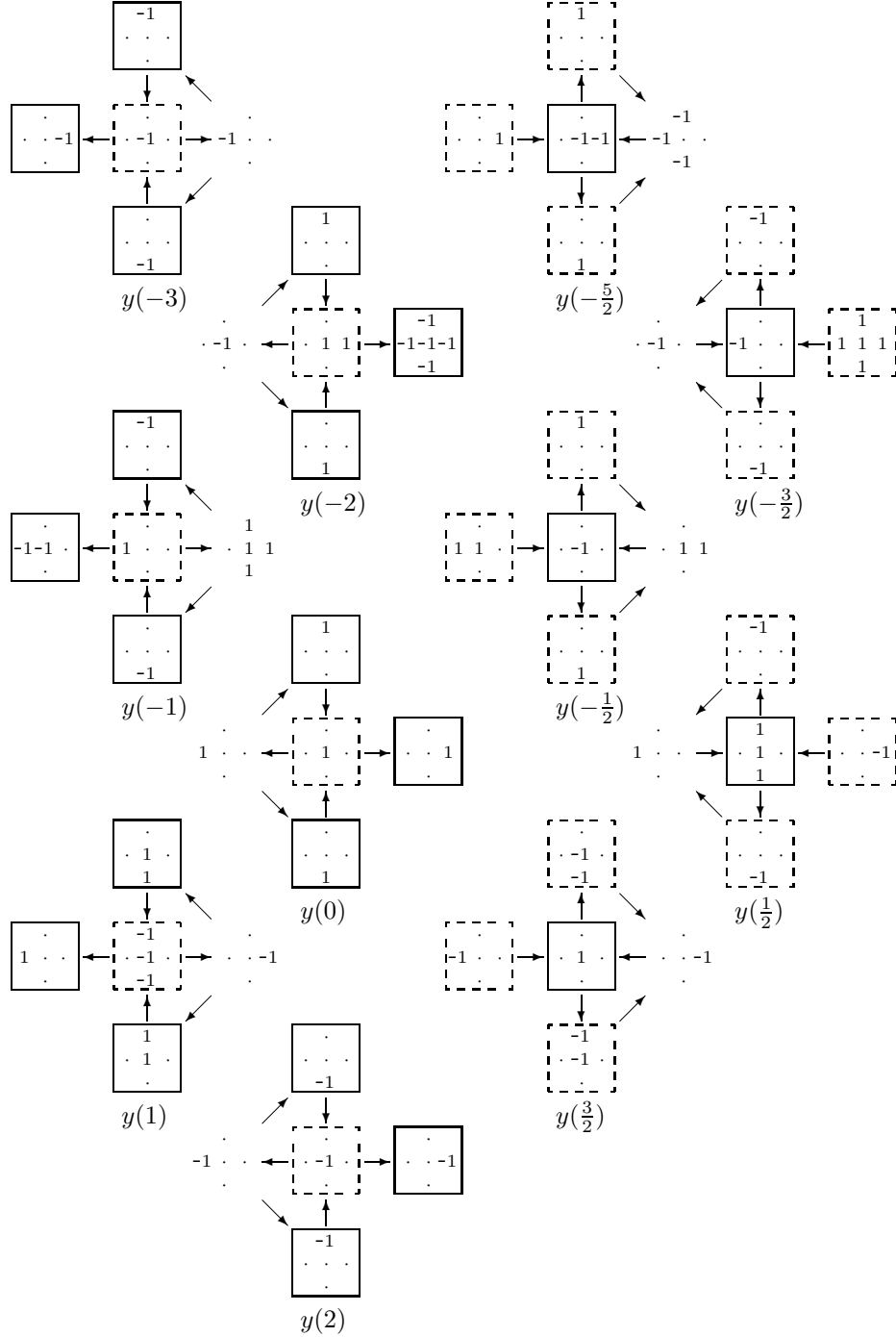+ & - & + & - & & + & - & + & -
\end{array}
\quad r\text{: odd}
$$

FIGURE 6. Tropical Y-system of type B_2 at level 2. We continue to use the convention in Figures 2–5 such that the variables framed by solid/dashed lines satisfy the condition $\mathbf{p}_+/\mathbf{p}_-$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_{2r-1}\}$, $-\Pi$, Φ_+ be the set of the simple roots, the negative simple roots, the positive roots, respectively, of type A_{2r-1}. Following [FZ2], we introduce the *piecewise-linear analogue* σ_i of the simple reflection s_i, acting on the set of the *almost positive roots* $\Phi_{\geq -1} = \Phi_+ \sqcup (-\Pi)$, by

$$\sigma_i(\alpha) = s_i(\alpha), \quad \alpha \in \Phi_+,$$

(3.2)
$$\sigma_i(-\alpha_j) = \begin{cases} \alpha_j & j = i, \\ -\alpha_j & \text{otherwise.} \end{cases}$$

Let

(3.3)
$$\sigma_+ = \prod_{i \in J_+} \sigma_i, \quad \sigma_- = \prod_{i \in J_-} \sigma_i,$$

where $J_\pm$ is the set of the vertices of A_{2r-1} with property $\pm$. We define σ as the composition

(3.4)
$$\sigma = \sigma_r \sigma_- \sigma_r \sigma_+.$$

Let $\omega : i \to 2r - i$ be an involution on J.

Lemma 3.4. *The following facts hold.*
(i) *For $i \in J_+$, $\sigma^k(-\alpha_i) \in \Phi_+$, $(1 \leq k \leq r-1)$, $\sigma^r(-\alpha_i) = -\alpha_{\omega(i)}$.*
(ii) *For $i \in J_-$, $\sigma^k(-\alpha_i) \in \Phi_+$, $(1 \leq k \leq r)$, $\sigma^{r+1}(-\alpha_i) = -\alpha_{\omega(i)}$.*
(iii) *$\sigma^k(-\alpha_r) \in \Phi_+$, $(1 \leq k \leq r-1)$, $\sigma^r(-\alpha_r) = -\alpha_r$.*
(iv) *$\sigma^k(\alpha_r) \in \Phi_+$, $(0 \leq k \leq r-1)$, $\sigma^r(\alpha_r) = \alpha_r$.*
(v) *The elements in Φ_+ in (i)–(iv) exhaust the set Φ_+, thereby providing the orbit decomposition of Φ_+ by σ.*

Proof. (i)–(iv). They are verified by explicitly calculating $\sigma^k(-\alpha_i)$ $(i \in J)$ and $\sigma^k(\alpha_r)$. The example for $r = 6$ is given in Table 1, where we use the notation

(3.5)
$$[i, j] = \alpha_i + \cdots + \alpha_j \quad (i < j), \quad [i] = \alpha_i.$$

In fact, it is not difficult to read off the general rule from this example.

(v). The total number of the elements in Φ_+ in (i)–(iv) is $r(2r-1)$, which coincides with $|\Phi_+|$. □

The orbit of $\sigma(-\alpha_i)$ $(i \neq r)$ is further described by the *root system of type A_{2r-2}*.

Lemma 3.5. *Let $O_i = \{\sigma^k(-\alpha_i) \mid 1 \leq k \leq r-1\}$ for $i \in J_+$ and $O_i = \{\sigma^k(-\alpha_i) \mid 1 \leq k \leq r\}$ for $i \in J_-$. Let Φ'_+ be the set of the positive roots of type A_{2r-2} with index set $J' = J - \{r\}$, and*

(3.6)
$$\rho : \bigsqcup_{i \in J'} O_i \to \Phi'_+$$

be the map which removes α_r from α if α contains α_r and does nothing otherwise. Then, ρ is a bijection, and its inverse $\rho^{-1}(\alpha')$ adds α_r if α' contains α_{r-1} and does nothing otherwise. Furthermore, under the bijection ρ, the action of σ is translated into the one of the Coxeter element $s = s_- s_+$ of type A_{2r-2} acting on Φ'_+, where $s_\pm = \prod_{i \in J_\pm} s_i$.

		-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11		
$1-$	$-\alpha_1$	[1]		[2,3]		[4,6]		[7,8]		[9,10]		[11]		$-\alpha_{11}$
$2+$	$-\alpha_2$		[1,3]		[2,6]		[4,8]		[7,10]		[9,11]		$-\alpha_{10}$	
$3-$	$-\alpha_3$	[3]		[1,6]		[2,8]		[4,10]		[7,11]		[9]		$-\alpha_9$
$4+$	$-\alpha_4$		[3,6]		[1,8]		[2,10]		[4,11]		[7,9]		$-\alpha_8$	
$5-$	$-\alpha_5$	[5,6]		[3,8]		[1,10]		[2,11]		[4,9]		[7]		$-\alpha_7$
6	$-\alpha_6$	[6] [5]	[6,8]	[3,5]	[6,10]	[1,5]	[6,11]	[2,5]	[6,9]	[4,5]	[6,7]		$-\alpha_6$ α_6	
$7+$	$-\alpha_7$		[5,8]		[3,10]		[1,11]		[2,9]		[4,7]		$-\alpha_5$	
$8-$	$-\alpha_8$	[8]		[5,10]		[3,11]		[1,9]		[2,7]		[4]		$-\alpha_4$
$9+$	$-\alpha_9$		[8,10]		[5,11]		[3,9]		[1,7]		[2,4]		$-\alpha_3$	
$10-$	$-\alpha_{10}$	[10]		[8,11]		[5,9]		[3,7]		[1,4]		[2]		$-\alpha_2$
$11+$	$-\alpha_{11}$		[10,11]		[8,9]		[5,7]		[3,4]		[1,2]		$-\alpha_1$	

TABLE 1. The orbits of $\sigma^k(-\alpha_i)$ and $\sigma^k(\alpha_r)$ in Φ_+ by σ of (3.4) for $r = 6$. The orbits of $-\alpha_6$ and α_6, i.e., $-\alpha_6 \to$ $[5] \to [3,5] \to \cdots \to -\alpha_6$ and $\alpha_6 \to [6,8] \to [6,10] \to \cdots \to \alpha_6$ are alternatively aligned. The numbers $-1, -2, \cdots$ in the head line will be identified with the parameter u in (3.7).

For $-h^\vee \leq u < 0$, define

$$
(3.7) \qquad \alpha_i(u) = \begin{cases} \sigma^{-u/2}(-\alpha_i) & i \in J_+, \ u \equiv 0, \\ \sigma^{-(u-1)/2}(-\alpha_i) & i \in J_-, \ u \equiv -1, \\ \sigma^{-(2u-1)/4}(-\alpha_r) & i = r, \ u \equiv -\frac{3}{2}, \\ \sigma^{-(2u+1)/4}(\alpha_r) & i = r, \ u \equiv -\frac{1}{2}, \end{cases}
$$

where $\equiv$ is modulo $2\mathbb{Z}$. Note that they correspond to the positive roots in Table 1 with u being the parameter in the head line. By Lemma 3.4 they are all the positive roots of A_{2r-1}.

Lemma 3.6. *The family in* (3.7) *satisfies the recurrence relations*

$$
(3.8) \qquad \begin{aligned} \alpha_i(u-1) + \alpha_i(u+1) &= \alpha_{i-1}(u) + \alpha_{i+1}(u) \quad (i \neq r-1, r, r+1), \\ \alpha_{r-1}(u-1) + \alpha_{r-1}(u+1) &= \alpha_{r-2}(u) + \alpha_{r+1}(u), \\ \alpha_{r+1}(u-1) + \alpha_{r+1}(u+1) &= \alpha_{r-1}(u) + \alpha_{r+2}(u), \\ \alpha_r(u - \tfrac{1}{2}) + \alpha_r(u + \tfrac{1}{2}) &= \alpha_{r-1}(u) \quad (u\text{: odd}), \\ \alpha_r(u - \tfrac{1}{2}) + \alpha_r(u + \tfrac{1}{2}) &= \alpha_{r+1}(u) \quad (u\text{: even}), \end{aligned}
$$

where $\alpha_0(u) = \alpha_{2r}(u) = 0$.

Proof. These relations are easily verified by the explicit expressions of $\alpha_i(u)$. See Table 1. The first three relations are also obtained from Lemma 3.5 and [FZ3, Eq. (10.9)]. $\qquad \square$

Let us return to prove (ii) of Proposition 3.2 for the powers of variables y_{i1} $(i \neq r)$ and y_{r2}. For a monomial m in $y = (y_\mathbf{i})_{\mathbf{i} \in \mathbf{I}}$, let $\pi_A(m)$ denote the specialization with $y_{r1} = y_{r3} = 1$. For simplicity, we set $y_{i1} = y_i$ $(i \neq r)$, $y_{r2} = y_r$, and also, $y_{i1}(u) = y_i(u)$ $(i \neq r)$, $y_{r2}(u) = y_r(u)$. We define the vectors $\mathbf{t}_i(u) = (t_i(u)_k)_{k=1}^{2r-1}$ by

$$
(3.9) \qquad \pi_A([y_i(u)]_\mathbf{T}) = \prod_{k=1}^{2r-1} y_k^{t_i(u)_k}.
$$

We also identify each vector $\mathbf{t}_i(u)$ with $\alpha = \sum_{k=1}^{2r-1} t_i(u)_k \alpha_k \in \mathbb{Z}\Pi$.

Proposition 3.7. *Let* $-h^\vee \leq u < 0$. *Then, we have*

$$
(3.10) \qquad \mathbf{t}_i(u) = -\alpha_i(u)
$$

for (i, u) *in* (3.7), *and*

$$
(3.11) \qquad \pi_A([y_{r1}(u)]_\mathbf{T}) = \pi_A([y_{r3}(u)]_\mathbf{T}) = 1, \quad u \equiv 0 \bmod 2\mathbb{Z}.
$$

Note that these formulas determine $\pi_A([y_\mathbf{i}(u)]_\mathbf{T})$ for any $(\mathbf{i}, u) : \mathbf{p}_+$.

Proof. We can verify the claim for $-2 \leq u \leq -\frac{1}{2}$ by direct computation. Then, by induction on u in the backward direction, one can establish the claim, together

with the recurrence relations among $\mathbf{t}_i(u)$'s with (i, u) in (3.7),

$$\mathbf{t}_i(u-1) + \mathbf{t}_i(u+1) = \mathbf{t}_{i-1}(u) + \mathbf{t}_{i+1}(u), \quad i \neq r-1, r, r+1,$$

$$\mathbf{t}_{r-1}(u-1) + \mathbf{t}_{r-1}(u+1) = \mathbf{t}_{r-2}(u) + \mathbf{t}_r(u - \tfrac{1}{2}) + \mathbf{t}_r(u + \tfrac{1}{2}),$$

$$(3.12) \qquad \mathbf{t}_{r+1}(u-1) + \mathbf{t}_{r+1}(u+1) = \mathbf{t}_{r+2}(u) + \mathbf{t}_r(u - \tfrac{1}{2}) + \mathbf{t}_r(u + \tfrac{1}{2}),$$

$$\mathbf{t}_r(u - \tfrac{1}{2}) + \mathbf{t}_r(u + \tfrac{1}{2}) = \mathbf{t}_{r-1}(u), \quad u\text{: odd},$$

$$\mathbf{t}_r(u - \tfrac{1}{2}) + \mathbf{t}_r(u + \tfrac{1}{2}) = \mathbf{t}_{r+1}(u), \quad u\text{: even}.$$

Note that (3.12) coincides with (3.8) under (3.10). To derive (3.12), one uses the mutations as in Figure 6 (or the tropical version of the Y-system $\mathbb{Y}_2(B_r)$ directly) and the positivity/negativity of $\pi_A([y_{\mathbf{i}}(u)]_{\mathbf{T}})$ resulting from (3.10) and (3.11) by induction hypothesis. $\qquad\square$

Now (ii) in Proposition 3.2 is an immediate consequence of Lemma 3.4 and Proposition 3.7. Finally, let us prove (iv), i.e., $\mathbf{t}_i(-h^\vee) = -\alpha_{\omega(i)}$. This is shown by the following formulas obtained from Lemma 3.4 and (2.19):

$$(3.13) \qquad \mathbf{t}_i(-h^\vee + 1) = \begin{cases} -\alpha_{\omega(i)-1} - \alpha_{\omega(i)} - \alpha_{\omega(i)+1} & i \neq r+1, \\ -\alpha_{r-2} - \alpha_{r-1} - \alpha_r - \alpha_{r+1} & i = r+1, \end{cases} \quad i \in J_+,$$

$$\mathbf{t}_i(-h^\vee + \tfrac{1}{2}) = \alpha_{\omega(i)}, \quad \mathbf{t}_i(-h^\vee) = -\alpha_{\omega(i)}, \quad i \in J_-,$$

$$\mathbf{t}_r(-h^\vee + 1) = \alpha_r + \alpha_{r+1}, \quad \mathbf{t}_r(-h^\vee + \tfrac{1}{2}) = -\alpha_r - \alpha_{r+1},$$

where $\alpha_0 = \alpha_{2r} = 0$. This completes the proof of Proposition 3.2.

4. Tropical Y-systems at higher levels

We have a generalization of Proposition 3.2 for the tropical Y-systems at higher levels.

Proposition 4.1. *Let $\ell > 2$ be an integer. For $[\mathcal{G}_Y(B, y)]_{\mathbf{T}}$ with $B = B_\ell(B_r)$, the following facts hold.*

(i) Let u be in the region $0 \leq u < \ell$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is positive.

(ii) Let u be in the region $-h^\vee \leq u < 0$.

(a) Let $\mathbf{i} \in \mathbf{I}^\circ \sqcup \mathbf{I}^\bullet_-$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative.

(b) Let $\mathbf{i} \in \mathbf{I}^\bullet_+$. For any $(\mathbf{i}, u) : \mathbf{p}_+$, the monomial $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ is negative for $u = -1, -3, \ldots$ and positive for $u = -2, -4, \ldots$.

(iii) $y_{ii'}(\ell) = y^{-1}_{i, \ell - i'}$ if $i \neq r$ and $y^{-1}_{r, 2\ell - i'}$ if $i = r$.

(iv) $y_{ii'}(-h^\vee) = y^{-1}_{2r-i, i'}$.

Proof. This proposition is a consequence of the *factorization property* of the tropical Y-system found in [Nkn] for simply laced case. Roughly speaking, in the region $-h^\vee \leq u \leq 0$, the system is factorized into the 'level 2 pieces', while in the region $0 \leq u \leq \ell$, the system is factorized into the 'type A pieces'.

First, we consider the region $-h^\vee \leq u < 0$. Let us concentrate on the simplest nontrivial example B_2 with level 3. It turns out that this example is almost general enough. In Figure 7, all the variables $[y_{\mathbf{i}}(u)]_{\mathbf{T}}$ in the region $-2 \leq u \leq 0$ are given explicitly. One had better view it in the backward direction from $u = 0$ to -2. If we look at only the first three rows from the bottom, we observe that the mutations occur in exactly the same pattern as the level 2 case in Figure 6. So is for the last
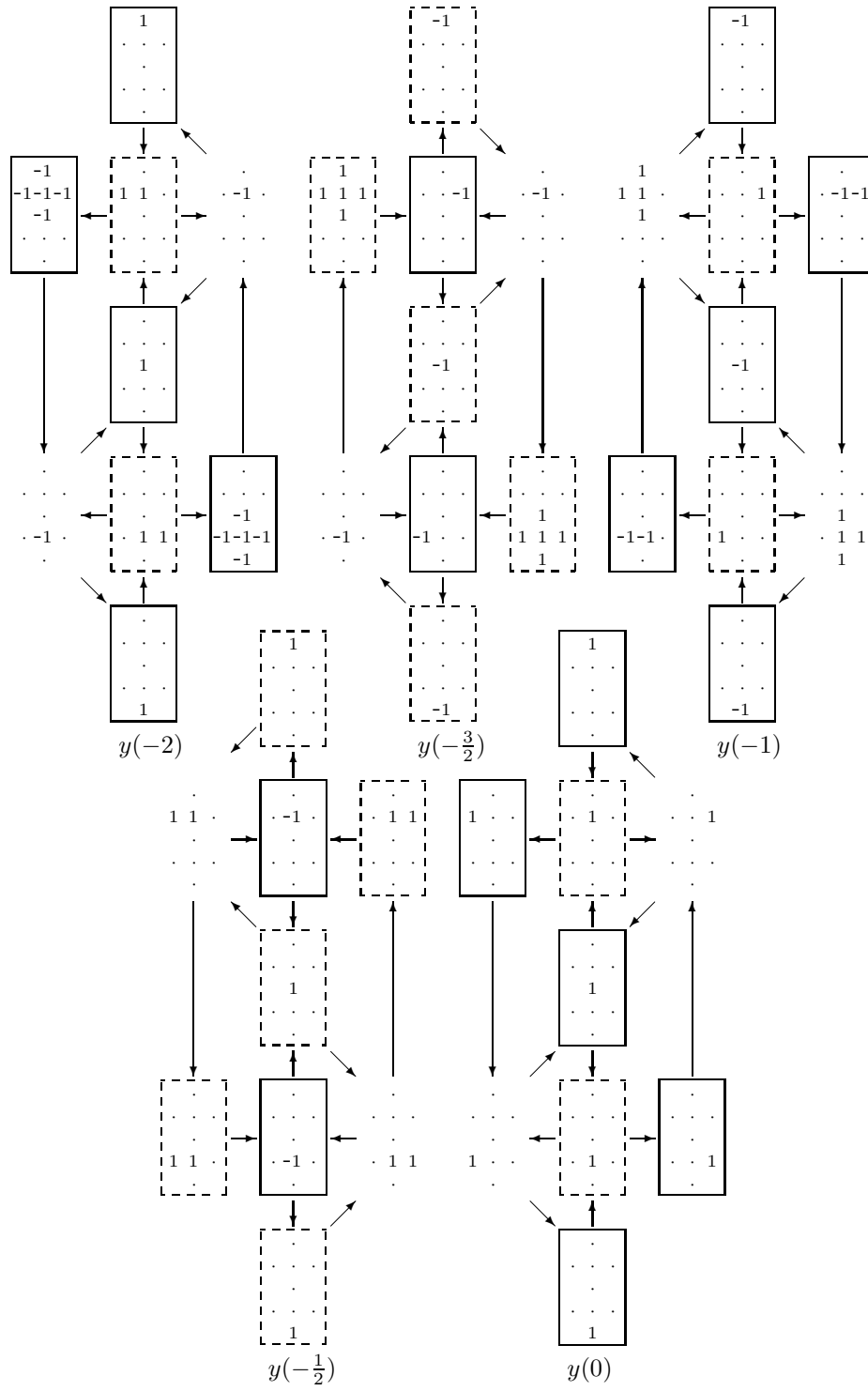
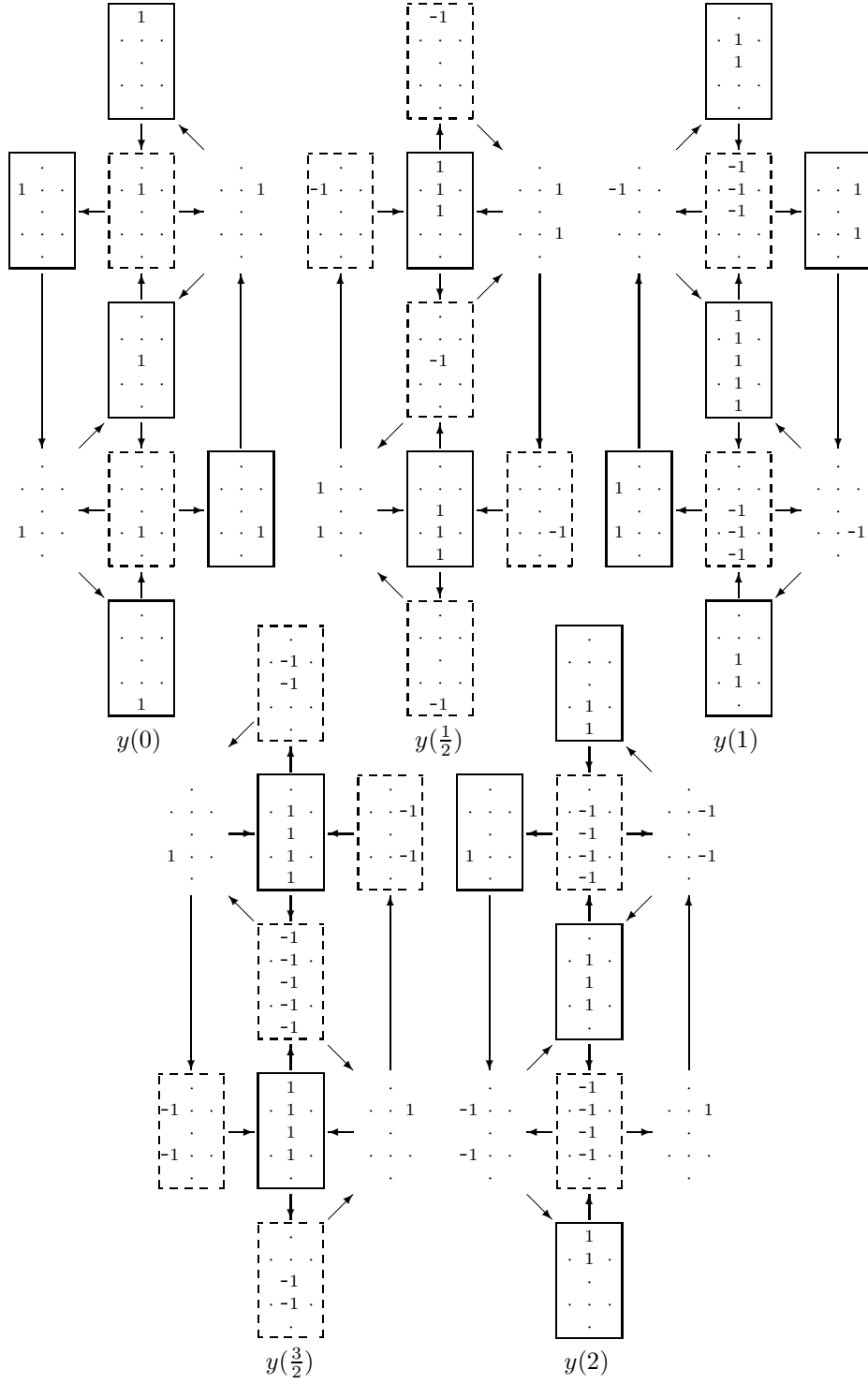FIGURE 7. Tropical Y-system of type B_2 at level 3 in the region $-2 \le u \le 0$.

FIGURE 8. Tropical Y-system of type B_2 at level 3 in the region $0 \leq u \leq 2$.

three rows (with 180° rotation). This is the factorization property. It occurs due to the coordination of the mutation sequence (2.14) and the positivity/negativity in Proposition 3.2. For example, in Figure 7, let us look at the vertical arrows between the second and the fourth rows. At any mutation point $(\mathbf{i}, u) : \mathbf{p}_-$ these arrows are *incoming* while $[y_\mathbf{i}(u)]_\mathbf{T}$ are *positive* by (2.19) and Proposition 3.2 (ii). Therefore, by Lemma 3.1, these arrows can be forgotten during mutations. By a similar reason, the variables in the third rows are not affected by mutations in the second and the fourth rows. Therefore, as long as the positivity in the second and the fourth rows continue, so does the factorization; hence (ii) holds. Moreover, (iv) holds because it does for level 2. This argument is also applicable to any rank r and level ℓ because of the definitions of the quiver $Q_\ell(B_r)$ and the mutation sequence (2.14).

Next, we consider the region $0 \le u < \ell$. Again, let us concentrate on the case B_2 with level 3. In Figure 8, all the variables $[y_\mathbf{i}(u)]_\mathbf{T}$ in the region $0 \le u \le 2$ are given explicitly. One had better view it in the forward direction from $u = 0$ to 2. We observe that each column does not affect each other in mutations. More precisely, the first and third columns (from the left) are in the same mutation pattern as the tropical Y-system of type A_2 studied in [FZ2, FZ3] (see also [Nkn] for more detail). The second column is in the same mutation pattern as the tropical Y-system of type A_5 (at the *twice* faster pace). Due to [FZ3, Proposition 10.7], $[y_\mathbf{i}(u)]_\mathbf{T}$ for $(\mathbf{i}, u) : \mathbf{p}_+$ are positive in the region $0 \le u < 3$ (3 is the Coxeter number of A_2 *and* the *half* of the Coxeter number of A_5). The factorization occurs by the same reason as before; at any mutation point $(\mathbf{i}, u) : \mathbf{p}_+$ the arrows between adjacent columns are *incoming* while $[y_\mathbf{i}(u)]_\mathbf{T}$ are *positive*. Therefore, these arrows can be forgotten, and we have (ii). Finally, (iii) is also a consequence of [FZ3, Proposition 10.7]. Again, this argument is also applicable to any rank r and level ℓ by replacing the number 3 in the above with ℓ, which is the Coxeter number of $A_{\ell-1}$ and the half of the Coxeter number of $A_{2\ell-1}$. $\square$

We obtain two important corollaries of Propositions 3.2 and 4.1.

Theorem 4.2. *For $[\mathcal{G}_Y(B, y)]_\mathbf{T}$ with $B = B_\ell(B_r)$, the following relations hold.*
  (i) Half periodicity: $[y_\mathbf{i}(u + h^\vee + \ell)]_\mathbf{T} = [y_{\boldsymbol{\omega}(\mathbf{i})}(u)]_\mathbf{T}$.
  (ii) Full periodicity: $[y_\mathbf{i}(u + 2(h^\vee + \ell))]_\mathbf{T} = [y_\mathbf{i}(u)]_\mathbf{T}$.

Proof. (i) follows from (iii) and (iv) of Propositions 3.2 and 4.1. (ii) follows from (i). $\square$

We remark that the half periodicity above is compatible with the one for mutation matrices; namely, set $B(u) = B := B_\ell(B_r), -B, \boldsymbol{r}(B), -\boldsymbol{r}(B)$ for $u \equiv 0, \frac{1}{2}, 1, \frac{3}{2}$ mod $2\mathbb{Z}$, respectively. Then,

$$(4.1) \qquad\qquad B(u + h^\vee + \ell) = \boldsymbol{\omega}(B(u))$$

holds due to Lemma 2.1 (ii).

Theorem 4.3. *For $[\mathcal{G}_Y(B, y)]_\mathbf{T}$ with $B = B_\ell(B_r)$, let N_+ and N_- denote the total numbers of the positive and negative monomials, respectively, among $[y_\mathbf{i}(u)]_\mathbf{T}$ for $(\mathbf{i}, u) : \mathbf{p}_+$ in the region $0 \le u < 2(h^\vee + \ell)$. Then, we have*

$$(4.2) \qquad\qquad N_+ = 2\ell(\ell r + \ell - 1), \quad N_- = 2r(2\ell r - 2r + 1).$$

Proof. This follows from (i) and (ii) of Propositions 3.2 and 4.1, and (i) in Theorem 4.2. $\square$

5. 2-Calabi-Yau realization and periodicity

In this section we prove Theorems 1.5 and 1.6. Our method is based on the 2-Calabi-Yau realizations of cluster algebras. They are triangulated categories satisfying 2-Calabi-Yau property, and their categorical structure realizes the combinatorial structure of cluster algebras.

5.1. **Periodicity theorem for cluster algebras.** In this subsection we assume that $B = (B_{ij})_{i,j \in I}$ is an arbitrary *skew-symmetric* (integer) matrix. Let $\mathcal{A}(B, x, y)$ be the cluster algebra with coefficients in the *universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y)$, where (B, x, y) is the initial seed. Alternatively, we may consider the *cluster pattern* assigned to the n-regular tree $\mathbb{T}_n$, $n = |I|$, with the following data (see Section 2.1). Let Q be the quiver (without loops or 2-cycles) corresponding to B with the set of vertices I. We fix a vertex $t_0 \in \mathbb{T}_n$, and assign the initial seed to t_0, $(Q(t_0), x(t_0), y(t_0)) = (Q, x, y)$. Then we have a seed $(Q(t), x(t), y(t))$ for each $t \in \mathbb{T}_n$, where we identify the quiver $Q(t)$ and the corresponding skew-symmetric matrix $B(t)$ in each seed.

Let $\mathrm{Trop}(y)$ be the tropical semifield. As in Section 3.1, we define the tropical evaluation of $y_i(t)$

$$(5.1) \qquad\qquad\qquad [y_i(t)]_{\mathbf{T}} \in \mathrm{Trop}(y)$$

as the image of $y_i(t) \in \mathbb{Q}_{\mathrm{sf}}(y)$ under the natural map $\pi_{\mathbf{T}} : \mathbb{Q}_{\mathrm{sf}}(y) \to \mathrm{Trop}(y)$. Here we call $[y_i(t)]_{\mathbf{T}}$'s the *principal coefficients* in accordance with the nomenclature of [FZ3]. (They may be also called the *tropical Y-variables*.)

For an automorphism $\omega : I \to I$ of I, we define a new quiver $\omega(Q)$ with the same set I of vertices by drawing an arrow $\omega(a) : \omega(i) \to \omega(j)$ in $\omega(Q)$ for each arrow $a : i \to j$ in Q.

The following theorem is Observation 1 explained in Section 1.5. While it is crucial in our proof of the periodicities of the T and Y-systems in this paper and also in [IIKKN], it is also expected to be useful to other applications.

Theorem 5.1 (Periodicity theorem). *Let B be an arbitrary skew-symmetric matrix, and let Q be the quiver corresponding to B. Let $(Q(t), x(t), y(t))$ be the seed at $t \in \mathbb{T}_n$ for $\mathcal{A}(B, x, y)$ as above. Suppose that there exists some $t \in \mathbb{T}_n$ and an automorphism ω of I such that $[y_i(t)]_{\mathbf{T}} = [y_{\omega(i)}]_{\mathbf{T}}$ holds for any $i \in I$. Then we have*

$$(5.2) \qquad \begin{aligned} &Q(t) = \omega^{-1}(Q) \quad (\textit{equivalently, } B_{ij}(t) = B_{\omega(i)\omega(j)}), \\ &x_i(t) = x_{\omega(i)} \quad (i \in I), \\ &y_i(t) = y_{\omega(i)} \quad (i \in I). \end{aligned}$$

In particular, the periodicity of seeds of $\mathcal{A}(B, x, y)$ coincides with the periodicity of principal coefficient tuples.

Let $\mathcal{A}_\bullet(B, x, y)$ be the cluster algebra with coefficients in the *tropical semifield* $\mathrm{Trop}(y)$, where (B, x, y) is the initial seed. It is also called the cluster algebra with *principal coefficients* [FZ3]. Note that a coefficient $y_i(t)$ in $\mathcal{A}_\bullet(B, x, y)$ coincides with $[y_i(t)]_{\mathbf{T}}$ in (5.1).

According to [FZ3, Theorem 4.6] and its proof, Theorem 5.1 reduces to the following result for $\mathcal{A}_\bullet(B, x, y)$.

Theorem 5.2. *Let B and Q be the same as in Theorem 5.1. Let $(Q(t), x(t), y(t))$ be the seed at $t \in \mathbb{T}_n$ for $\mathcal{A}_\bullet(B, x, y)$. Suppose that there exists some $t \in \mathbb{T}_n$ and an automorphism ω of I such that $y_i(t) = y_{\omega(i)}$ holds for any $i \in I$. Then, we have*

(5.3)
$$Q(t) = \omega^{-1}(Q),$$
$$x_i(t) = x_{\omega(i)} \quad (i \in I).$$

We give a proof of Theorem 5.2 in Section 5.3.

Remark 5.3. Theorem 5.2 was essentially conjectured by Fomin-Zelevinsky [FZ3, Conjecture 4.7] for an *arbitrary skew-symmetrizable* matrix B. Therefore, we partly prove the conjecture for a *skew-symmetric* matrix B. Note that our claim is a little stronger than [FZ3, Conjecture 4.7] because the periodicity of the *principal parts* of exchange matrices also follows from the periodicity of principal coefficients. (Meanwhile, the periodicity of the *complementary parts* of exchange matrices coincides with the periodicity of principal coefficients by definition [FZ3].)

Remark 5.4. Since Theorems 5.1 and 5.2 involve quivers corresponding to arbitrary skew-symmetric matrices, the categories appearing in their proof will in general have infinite-dimensional morphism spaces. It is an interesting question to ask whether the quivers which appear in our applications of the two theorems actually admit categorifications by Hom-finite 2-CY categories. We conjecture that this is indeed the case for all levels $l \geq 2$. For $l = 2$, it is clear since the quiver $Q_2(B_r)$ is of finite cluster type D_{2r+1}. However, for $l > 2$, the conjecture appears to be nontrivial.

5.2. **2-Calabi-Yau realization of $\mathcal{A}_\bullet(B, x, y)$.** Our proof of Theorem 5.2 uses the categorification of the cluster algebra $\mathcal{A}_\bullet(B, x, y)$ by a certain 2-Calabi-Yau category [Pa, FK, Ke1, A, KY, Pl1, Pl2]. Here we review the recent result by Plamondon [Pl1, Pl2].

Let Q be the quiver corresponding to an arbitrary skew-symmetric matrix B. Define the *principal extension* $\widetilde{Q}$ of Q as the quiver obtained from Q by adding a new vertex i' and an arrow $i' \to i$ for each $i \in I$. Thus the set of vertices in $\widetilde{Q}$ is given by $\widetilde{I} := I \sqcup I'$ with $I' := \{i' \mid i \in I\}$. By mutations one can associate a quiver $\widetilde{Q}(t)$ with each $t \in \mathbb{T}_n$, where $\widetilde{Q}(t)$ contains $Q(t)$ as a full subquiver. Note that we do not make mutations for 'frozen indices' $i' \in I'$.

We fix a base field K to be an infinite one. Since $\widetilde{Q}$ does not have loops and 2-cycles, we have the following result by [DWZ1, Corollary 7.4].

Proposition 5.5. *There exists a non-degenerate potential W on $\widetilde{Q}$.*

From now on we assume W is a non-degenerate potential on $\widetilde{Q}$. We denote by

(5.4)
$$\mathcal{C} := \mathcal{C}_{(\widetilde{Q}, W)}$$

the *cluster category* associated to the quiver with potential $(\widetilde{Q}, W)$, which is not necessarily Hom-finite [A, KY, Pl1]. The category $\mathcal{C}$ canonically contains a rigid object

(5.5)
$$T = \bigoplus_{i \in \widetilde{I}} T_i \in \mathcal{C}$$

such that $\mathrm{End}_{\mathcal{C}}(T)$ is isomorphic to the Jacobian algebra of $(\widetilde{Q}, W)$. For each $t \in \mathbb{T}_n$, we have a rigid object

$$(5.6) \qquad T(t) = \bigoplus_{i \in \widetilde{I}} T_i(t) \in \mathcal{C}$$

by applying successive mutations (see [Pl1, Section 2.6]). We have $T_{i'}(t) = T_{i'}$ for any $i' \in I'$.

From the definition of $T(t)$ and the non-degeneracy of $(\widetilde{Q}, W)$ we have the following description of $\widetilde{Q}(t)$.

Proposition 5.6. *For each $t \in \mathbb{T}_n$, the quiver of $\mathrm{End}_{\mathcal{C}}(T(t))$ is $\widetilde{Q}(t)$, where each vertex $i \in \widetilde{I}$ corresponds to the direct summand $T_i(t)$ of $T(t)$.*

As usual we denote by $\mathrm{add}\, T(t)$ the full subcategory of $\mathcal{C}$ consisting of all direct summands of finite direct sums of copies of $T(t)$. We denote by

$$(5.7) \qquad \mathrm{pr}\, T(t)$$

the full subcategory of $\mathcal{C}$ consisting of objects $M \in \mathcal{C}$ such that there exists a triangle

$$(5.8) \qquad T'' \to T' \to M \to T''[1]$$

in $\mathcal{C}$ with $T', T'' \in \mathrm{add}\, T(t)$.

Proposition 5.7 ([Pl1, Proposition 2.7, Corollary 2.12]). *(1) We have $\mathrm{pr}\, T(t) = \mathrm{pr}\, T$ for any $t \in \mathbb{T}_n$.*

(2) The category $\mathrm{pr}\, T$ is Krull-Schmidt in the sense that any object can be written as a finite direct sum of objects whose endomorphism rings are local.

Now let us introduce the following notion.

Definition 5.8. Let $t \in \mathbb{T}_n$. For an object $T' = \bigoplus_{i \in \widetilde{I}} T_i(t)^{\ell_i}$ in $\mathrm{add}\, T(t)$, we put

$$(5.9) \qquad [T']_{T(t)} := (\ell_i)_{i \in \widetilde{I}} \in \mathbb{Z}^{\widetilde{I}}.$$

For an object $M \in \mathrm{pr}\, T(t)$, we take a triangle (5.8) and define the *index* of M by

$$(5.10) \qquad \mathrm{ind}_{T(t)}(M) := [T']_{T(t)} - [T'']_{T(t)} \in \mathbb{Z}^{\widetilde{I}}.$$

This is independent of the choice of the triangle (5.8) by Proposition 5.7(2).

We have the following relationship between indices and principal coefficients.

Proposition 5.9 ([Pl2, Corollary 3.10], [Ke1, Theorem 7.13(b)]). *Let $y_i(t)$ be a coefficient in $\mathcal{A}_{\bullet}(B, x, y)$. For $t \in \mathbb{T}_n$, we put $y_j(t) = \prod_{i \in I} y_i^{c_{ij}(t)}$ for any $j \in I$. Then we have*

$$(5.11) \qquad -\mathrm{ind}_{T(t)}(T_i[1]) = (c_{ij}(t))_{j \in I} \in \mathbb{Z}^I$$

for any $i \in I$, where we embed $\mathbb{Z}^I$ into $\mathbb{Z}^{\widetilde{I}}$ naturally.

Remark 5.10. Proposition 5.9 shows that the *principal coefficients* of a quiver Q are determined by the *g-vectors* of the opposite quiver Q^{op}. This can also be deduced from Conjecture 1.6 (proved in Theorem 1.7) of [DWZ2] using Remark 7.15 of [FZ3] on 'Langlands duality'.

The following analogue of [DK, Theorem 2.3] is an important ingredient in our proof.

Proposition 5.11 ([Pl2, Proposition 3.1]). *Let $X, Y \in \operatorname{pr} T$ be rigid objects and $t \in \mathbb{T}_n$. Then $X \simeq Y$ if and only if $\operatorname{ind}_{T(t)}(X) = \operatorname{ind}_{T(t)}(Y)$.*

Let us introduce Caldero-Chapoton-type map.

Definition 5.12. Define a full subcategory

$$(5.12) \qquad \mathcal{D} := \{M \in \operatorname{pr} T \cap \operatorname{pr} T[-1] \mid \dim_K \operatorname{Hom}_{\mathcal{C}}(T, M[1]) < \infty\}.$$

For any object $M \in \mathcal{D}$, we define an element $\mathbb{X}_M$ in $\mathbb{Z}[x^{\pm 1}, y]$ by

$$(5.13) \qquad \widehat{y}_j := y_j \prod_{i \in I} x_i^{B_{ij}} \quad (j \in I),$$

$$(5.14) \qquad \mathbb{X}_M := \left(\prod_{i \in I} x_i^{\operatorname{ind}_T(M)_i} y_i^{\operatorname{ind}_T(M)_{i'}} \right) \sum_{e \in \mathbb{Z}^{\tilde{I}}} \chi(\operatorname{Gr}_e(\operatorname{Hom}_{\mathcal{C}}(T, M[1]))) \prod_{j \in I} \widehat{y}_j^{e_j},$$

where Gr_e is the quiver Grassmannian and χ is the Euler characteristic.

For each $t \in \mathbb{T}_n$ we have $T(t) \in \mathcal{D}$ by Proposition 5.7(1). The following description of cluster variables in $\mathcal{A}_\bullet(B, x, y)$ is crucial in our proof.

Proposition 5.13. *Let $x_i(t)$ be a cluster variable in $\mathcal{A}_\bullet(B, x, y)$. Then, we have*

$$(5.15) \qquad\qquad x_i(t) = \mathbb{X}_{T_i(t)}$$

for any $t \in \mathbb{T}_n$ and $i \in I$.

Proof. Specializing (B, n) in [Pl1, Theorem 3.12] to

$$(5.16) \qquad\qquad \left(\begin{pmatrix} B & -E_n \\ E_n & O \end{pmatrix}, 2n \right),$$

we have that $\mathbb{X}$ is a cluster character in the sense of [Pl1, Definition 3.10]. Now the assertion is an immediate consequence. $\qquad \square$

5.3. Proof of Theorem 5.2.
Now we are ready to prove Theorem 5.2.

Since $y_j(t) = y_{\omega(j)}$ for any $j \in I$, we have in Proposition 5.9

$$(5.17) \qquad\qquad c_{ij}(t) = \begin{cases} 1 & i = \omega(j) \\ 0 & \text{otherwise} \end{cases}$$

for any $i, j \in I$. Thus $\operatorname{ind}_{T(t)}(T_{\omega(j)}[1]) = \operatorname{ind}_{T(t)}(T_j(t)[1])$ for any $j \in I$. By Proposition 5.11 we have

$$(5.18) \qquad\qquad T_{\omega(j)} \simeq T_j(t)$$

for any $j \in I$. By Proposition 5.13 and (5.18), we have

$$(5.19) \qquad\qquad x_j(t) = \mathbb{X}_{T_j(t)} = \mathbb{X}_{T_{\omega(j)}} = x_{\omega(j)}$$

for any $j \in I$.

Finally by (5.18) and Proposition 5.6 we have

$$(5.20) \qquad\qquad \omega^{-1}(\widetilde{Q}) = \widetilde{Q}(t) \quad \text{and so} \quad \omega^{-1}(Q) = Q(t).$$

$$\square$$

5.4. **Proof of periodicities of T and Y-systems.** Now the proof of Theorems 1.5 and 1.6 is at hand.

As corollaries of Theorems 4.2 and 5.1 we immediately obtain the periodicities of cluster variables and coefficients in $\mathcal{A}(B, x, y)$ with $B = B_\ell(B_r)$.

Corollary 5.14. *For $\mathcal{A}(B, x, y)$ with $B = B_\ell(B_r)$, the following relations hold.*
(i) *Half periodicity:* $x_{\mathbf{i}}(u + h^\vee + \ell) = x_{\boldsymbol{\omega}(\mathbf{i})}(u)$.
(ii) *Full periodicity:* $x_{\mathbf{i}}(u + 2(h^\vee + \ell)) = x_{\mathbf{i}}(u)$.

Corollary 5.15. *For $\mathcal{G}(B, y)$ with $B = B_\ell(B_r)$, the following relations hold.*
(i) *Half periodicity:* $y_{\mathbf{i}}(u + h^\vee + \ell) = y_{\boldsymbol{\omega}(\mathbf{i})}(u)$.
(ii) *Full periodicity:* $y_{\mathbf{i}}(u + 2(h^\vee + \ell)) = y_{\mathbf{i}}(u)$.

As further corollaries of Corollaries 5.14 and 5.15 and Theorems 2.7 and 2.10 we obtain Theorems 1.5 and 1.6.

As a corollary of Theorems 4.2 and 5.2 we also obtain the periodicity of F-polynomials [FZ3] (see Section 2.1 for the definition), which will be used in the next section.

Corollary 5.16. *For $\mathcal{A}(B, x, y)$ with $B = B_\ell(B_r)$, let $F_{\mathbf{i}}(u)$ be the F-polynomial at $(\mathbf{i}, u)$. Then, the following relations hold.*
(i) *Half periodicity:* $F_{\mathbf{i}}(u + h^\vee + \ell) = F_{\boldsymbol{\omega}(\mathbf{i})}(u)$.
(ii) *Full periodicity:* $F_{\mathbf{i}}(u + 2(h^\vee + \ell)) = F_{\mathbf{i}}(u)$.

6. Dilogarithm identities

In this section we prove Theorem 1.10.

In the cluster algebraic formulation here, Theorem 1.10 is expressed as follows.

Theorem 6.1. *For $\mathcal{G}_Y(B, y)$ with $B = B_\ell(B_r)$, let $y_m^{(a)}(u)$ be the coefficient tuple in Theorem 2.10. Then, for any semifield homomorphism $\varphi : \mathbb{Q}_{\mathrm{sf}}(y) \to \mathbb{R}_+$, we have the identity*

$$(6.1) \qquad \frac{6}{\pi^2} \sum_{(a,m,u) \in S'_+} L\left(\frac{\varphi(y_m^{(a)}(u))}{1 + \varphi(y_m^{(a)}(u))} \right) = 2r(2r\ell - 2r + 1),$$

where $S'_+ = \{(a, m, u) \in \mathcal{I}'_{\ell+} \mid 0 \leq u < 2(h^\vee + \ell)\}$.

Let $F_m^{(a)}(u)$ denote the F-polynomial $F_{\mathbf{i}}(v)$ at $(\mathbf{i}, v) = g((a, m, u))$, i.e., with the same parametrization by $\mathcal{I}_{\ell+}$ as $x_m^{(a)}(u)$.

Lemma 6.2. *(i) For $(a, m, u) \in \mathcal{I}'_{\ell+}$, the following relations hold.*

(6.2)

$$F_m^{(a)}\left(u - \frac{1}{t_a}\right) F_m^{(a)}\left(u + \frac{1}{t_a}\right) = \left[\frac{y_m^{(a)}(u)}{1 + y_m^{(a)}(u)} \right]_{\mathbf{T}} \prod_{(b,k,v) \in \mathcal{I}_{\ell+}} F_k^{(b)}(v)^{G(b,k,v;\,a,m,u)}$$

$$+ \left[\frac{1}{1 + y_m^{(a)}(u)} \right]_{\mathbf{T}} F_{m-1}^{(a)}(u) F_{m+1}^{(a)}(u),$$

$$(6.3) \qquad y_m^{(a)}(u) = [y_m^{(a)}(u)]_{\mathbf{T}} \frac{\prod\limits_{(b,k,v)\in\mathcal{I}_{\ell+}} F_k^{(b)}(v)^{G(b,k,v;\,a,m,u)}}{F_{m-1}^{(a)}(u)F_{m+1}^{(a)}(u)},$$

$$(6.4) \qquad 1 + y_m^{(a)}(u) = [1 + y_m^{(a)}(u)]_{\mathbf{T}} \frac{F_m^{(a)}\left(u - \frac{1}{t_a}\right) F_m^{(a)}\left(u + \frac{1}{t_a}\right)}{F_{m-1}^{(a)}(u)F_{m+1}^{(a)}(u)}.$$

(ii) *Periodicity:* $F_m^{(a)}(u + 2(h^\vee + \ell)) = F_m^{(a)}(u)$.
(iii) *Each polynomial $F_m^{(a)}(u)$ has constant term 1.*

Proof. (i). (6.2) is a specialization of (2.4). (6.3) is due to [FZ3, Proposition 3.13]. (6.4) follows from (6.2) and (6.3). (ii). This is a special case of Corollary 5.16. (iii). The claim is shown by induction on u, by using $F_{\mathbf{i}}(0) = 1$, (6.2), and Proposition 4.1 (cf. [FZ3, Proposition 5.6]). $\qquad\square$

According to [FS, C, Nkn], the proof of Theorem 1.10 reduces to the next claim.

Proposition 6.3. *(i) In $\bigwedge^2 \mathbb{Q}_{\mathrm{sf}}(y)$, we have*

$$(6.5) \qquad \sum_{(a,m,u)\in S'_+} y_m^{(a)}(u) \wedge (1 + y_m^{(a)}(u)) = 0.$$

(ii) The total number of the negative monomials among $[y_m^{(a)}(u)]_{\mathbf{T}}$ ($(a,m,u) \in S'_+$) is $2r(2r\ell - 2r + 1)$.

(ii) is already proved in Theorem 4.3. Let us prove (i). It is parallel to the simply laced case [Nkn, Proposition 4.1], but little more complicated. Therefore, we present the calculations.

We put (6.3) and (6.4) into (6.5), and expand it.

Firstly,

$$(6.6) \qquad \sum_{(a,m,u)\in S'_+} [y_m^{(a)}(u)]_{\mathbf{T}} \wedge [1 + y_m^{(a)}(u)]_{\mathbf{T}} = 0,$$

since each monomial $[y_m^{(a)}(u)]_{\mathbf{T}}$ is either positive or negative by Proposition 4.1.

Secondly, the contributions from the terms involving only $F_{\mathbf{i}}(u)$'s vanish. To see it, we separate them into two parts. The first part

$$(6.7) \qquad \sum_{(a,m,u)\in S'_+} F_{m-1}^{(a)}(u)F_{m+1}^{(a)}(u) \wedge F_m^{(a)}(u - \tfrac{1}{t_a})F_m^{(a)}(u + \tfrac{1}{t_a})$$

vanishes due to the symmetry argument of [CGT, Section 3], where we use the periodicity of F-polynomials (Lemma 6.2 (ii)). The second part

$$(6.8) \qquad \sum_{(a,m,u)\in S'_+} \prod_{(b,k,v)\in\mathcal{I}_{\ell+}} F_k^{(b)}(v)^{G(b,k,v;\,a,m,u)} \wedge \frac{F_m^{(a)}\left(u - \frac{1}{t_a}\right) F_m^{(a)}\left(u + \frac{1}{t_a}\right)}{F_{m-1}^{(a)}(u)F_{m+1}^{(a)}(u)}$$

reduces, by the symmetry argument again, to the sum consisting of the terms with $(a, b) = (r - 1, r), (r, r - 1)$; namely,

(6.9)

$$
\sum_{m=1}^{\ell-1} \sum_{\substack{u \equiv 0 \bmod \mathbb{Z} \\ 0 \le u < 2(h^\vee + \ell)}} F_{2m}^{(r)}(u) \wedge \frac{F_m^{(r-1)}(u - 1) \, F_m^{(r-1)}(u + 1)}{F_{m-1}^{(r-1)}(u) F_{m+1}^{(r-1)}(u)}
$$

$$
+ \sum_{m=1}^{\ell-1} \sum_{\substack{u \equiv \frac{1}{2} \bmod \mathbb{Z} \\ 0 \le u < 2(h^\vee + \ell)}} F_m^{(r-1)}(u - \tfrac{1}{2}) F_m^{(r-1)}(u + \tfrac{1}{2}) \wedge \frac{F_{2m}^{(r)}\left(u - \frac{1}{2}\right) F_{2m}^{(r)}\left(u + \frac{1}{2}\right)}{F_{2m-1}^{(r)}(u) F_{2m+1}^{(r)}(u)}
$$

$$
+ \sum_{m=0}^{\ell-1} \sum_{\substack{u \equiv 0 \bmod \mathbb{Z} \\ 0 \le u < 2(h^\vee + \ell)}} F_m^{(r-1)}(u) F_{m+1}^{(r-1)}(u) \wedge \frac{F_{2m+1}^{(r)}\left(u - \frac{1}{2}\right) F_{2m+1}^{(r)}\left(u + \frac{1}{2}\right)}{F_{2m}^{(r)}(u) F_{2m+2}^{(r)}(u)},
$$

where $F_0^{(r-1)}(u) = F_\ell^{(r-1)}(u) = F_0^{(r)}(u) = F_{2\ell}^{(r)}(u) = 1$. It is easy to check that all the terms cancel each other.

Thirdly, the contribution from the remaining terms are as follows, where $S_+ = \{(a, m, u) \in \mathcal{I}_{\ell+} \mid 0 \le u < 2(h^\vee + \ell)\}$:

(6.10)

$$
\sum_{(a,m,u) \in S'_+} [y_m^{(a)}(u)]_{\mathbf{T}} \wedge F_m^{(a)}(u - \tfrac{1}{t_a}) F_m^{(a)}(u + \tfrac{1}{t_a})
$$

$$
= \sum_{(a,m,u) \in S_+} [y_m^{(a)}(u - \tfrac{1}{t_a})]_{\mathbf{T}} [y_m^{(a)}(u + \tfrac{1}{t_a})]_{\mathbf{T}} \wedge F_m^{(a)}(u),
$$

(6.11)

$$
- \sum_{(a,m,u) \in S'_+} [y_m^{(a)}(u)]_{\mathbf{T}} \wedge F_{m-1}^{(a)}(u) F_{m+1}^{(a)}(u)
$$

$$
= - \sum_{(a,m,u) \in S_+} [y_{m-1}^{(a)}(u)]_{\mathbf{T}} [y_{m+1}^{(a)}(u)]_{\mathbf{T}} \wedge F_m^{(a)}(u),
$$

(6.12)

$$
\sum_{(a,m,u) \in S'_+} [1 + y_m^{(a)}(u)]_{\mathbf{T}} \wedge F_{m-1}^{(a)}(u) F_{m+1}^{(a)}(u)
$$

$$
= \sum_{(a,m,u) \in S_+} [1 + y_{m-1}^{(a)}(u)]_{\mathbf{T}} [1 + y_{m+1}^{(a)}(u)]_{\mathbf{T}} \wedge F_m^{(a)}(u),
$$

(6.13)

$$
- \sum_{(a,m,u) \in S'_+} [1 + y_m^{(a)}(u)]_{\mathbf{T}} \wedge \prod_{(b,k,v) \in \mathcal{I}_{\ell+}} F_k^{(b)}(v)^{G(b,k,v;\,a,m,u)}
$$

$$
= - \sum_{(a,m,u) \in S_+} \prod_{(b,k,v) \in \mathcal{I}'_{\ell+}} [1 + y_k^{(b)}(v)]_{\mathbf{T}}^{G(a,m,u;\,b,k,v)} \wedge F_m^{(a)}(u).
$$

These terms cancel if we have the relation

(6.14)

$$
[y_m^{(a)}(u - \tfrac{1}{t_a})]_{\mathbf{T}} [y_m^{(a)}(u + \tfrac{1}{t_a})]_{\mathbf{T}} = \frac{\displaystyle\prod_{(b,k,v) \in \mathcal{I}'_{\ell+}} [1 + y_k^{(b)}(v)]_{\mathbf{T}}^{{}^t G(b,k,v;\,a,m,u)}}{[1 + y_{m-1}^{(a)}(u)^{-1}]_{\mathbf{T}} [1 + y_{m+1}^{(a)}(u)^{-1}]_{\mathbf{T}}}.
$$

This is nothing but the Y-system (1.6), therefore, satisfied by Lemma 2.8.

This completes the proof of Proposition 6.3.

7. Alternative proof of periodicities of T and Y-systems of simply laced type

Let $(X_r, X'_{r'})$ be a pair of simply laced Dynkin diagrams of finite type with index sets I and I'.

As an application of Theorem 5.1, we give an alternative and simplified proof of the periodicities of the T and Y-systems associated with $(X_r, X'_{r'})$. They were formerly proved by [FZ1, FZ2] for $X'_{r'} = A_1$ ('level 2 case') and [Ke1, IIKNS, Ke2] for general case.

For a family of variables $\{T_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$, the *T-system* $\mathbb{T}(X_r, X'_{r'})$ *associated with a pair* $(X_r, X'_{r'})$ *is a system of the relations*

$$(7.1) \qquad T_{ii'}(u-1)T_{ii'}(u+1) = \prod_{j:j\sim i} T_{ji'}(u) + \prod_{j':j'\sim i'} T_{ij'}(u),$$

where $j \sim i$ means j is adjacent to i in X_r, while $j' \sim i'$ means j' is adjacent to i' in $X'_{r'}$.

For a family of variables $\{Y_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$, the *Y-system* $\mathbb{Y}(X_r, X'_{r'})$ *associated with a pair* $(X_r, X'_{r'})$ *is a system of the relations*

$$(7.2) \qquad Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\displaystyle\prod_{j:j\sim i}(1 + Y_{ji'}(u))}{\displaystyle\prod_{j':j'\sim i'}(1 + Y_{ij'}(u)^{-1})}.$$

Let $C = (C_{ij})_{i,j\in I}$ and $C' = (C_{i'j'})_{i',j'\in I}$ be a pair of Cartan matrices of types X_r and $X'_{r'}$ with fixed bipartite decompositions $I = I_+ \sqcup I_-$ and $I' = I'_+ \sqcup I'_-$. Set $\mathbf{I} = I \times I'$. For $\mathbf{i} = (i, i') \in \mathbf{I}$, let us write $\mathbf{i} : (++)$ if $(i, i') \in I_+ \times I'_+$, *etc.* Define the matrix $B = B(X_r, X'_{r'}) = (B_{\mathbf{ij}})_{\mathbf{i,j}\in\mathbf{I}}$ by

$$(7.3) \qquad B_{\mathbf{ij}} = \begin{cases} -C_{ij}\delta_{i'j'} & \mathbf{i} : (-+), \mathbf{j} : (++) \text{ or } \mathbf{i} : (+-), \mathbf{j} : (--), \\ C_{ij}\delta_{i'j'} & \mathbf{i} : (++), \mathbf{j} : (-+) \text{ or } \mathbf{i} : (--), \mathbf{j} : (+-), \\ -\delta_{ij}C'_{i'j'} & \mathbf{i} : (++), \mathbf{j} : (+-) \text{ or } \mathbf{i} : (--), \mathbf{j} : (-+), \\ \delta_{ij}C'_{i'j'} & \mathbf{i} : (+-), \mathbf{j} : (++) \text{ or } \mathbf{i} : (-+), \mathbf{j} : (--), \\ 0 & \text{otherwise.} \end{cases}$$

Then, as in Section 2, one can formulate the T and Y-systems in terms of the cluster algebra $\mathcal{A}(B, x, y)$ and its coefficient group $\mathcal{G}(B, y)$ with $B = B(X_r, X'_{r'})$ (cf. [Nkn, Proposition 2.6]).

Theorem 7.1. *The following relations hold for the tropical Y-system of* $\mathcal{G}(B, y)$ *with* $B = B(X_r, X'_{r'})$.
  (i) Half periodicity: $[y_{\mathbf{i}}(u + h + h')]_{\mathbf{T}} = [y_{\boldsymbol{\omega}(\mathbf{i})}(u)]_{\mathbf{T}}$.
  (ii) Full periodicity: $[y_{\mathbf{i}}(u + 2(h + h'))]_{\mathbf{T}} = [y_{\mathbf{i}}(u)]_{\mathbf{T}}$.
  Here, h and h' are the Coxeter numbers of X_r and $X'_{r'}$, and $\boldsymbol{\omega} = \omega \times \omega'$, where ω (resp. ω') is the Dynkin automorphism of X_r (resp. $X'_{r'}$) for types A_r, D_r (r : odd), or E_6, and the identity otherwise.

Proof. This is an immediate consequence of the factorization property of the tropical Y-system studied in [Nkn, Proposition 3.2]. $\qquad\square$

As a corollary of Theorem 5.1 and Theorem 7.1, we obtain the periodicities of the T and Y-systems.

Corollary 7.2. *The following relations hold.*
  (i) Half periodicity: $T_{\mathbf{i}}(u + h + h') = T_{\boldsymbol{\omega}(\mathbf{i})}(u)$, $Y_{\mathbf{i}}(u + h + h') = Y_{\boldsymbol{\omega}(\mathbf{i})}(u)$.
  (ii) Full periodicity: $T_{\mathbf{i}}(u + 2(h + h')) = T_{\mathbf{i}}(u)$, $Y_{\mathbf{i}}(u + 2(h + h')) = Y_{\mathbf{i}}(u)$.

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